VII. *A Theory of Asymptotic Series.*

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Introduction and Historical Summary.

In their efforts to place mathematical analysis on the firmest possible foundations, Abel and Cauchy found it necessary to banish non-convergent series from their work; from that time until a quarter of a century ago the theory of divergent series was, in general, neglected by mathematicians.

A consistent theory of divergent series was, however, developed by Poincaré in 1886, and, ten years later, Borel enunciated his theory of summability in connection with oscillating series. So far as diverging power series are concerned, the theory of Borel is more precise than that of Poincaré.

Since the paper of **Poincaré** appeared, researches have been published by a host of mathematicians. It is sufficient to mention the names of **Cesàro**, **Le Roy**, **Van Vleck**, **Stieltjes**, **Mellin**, **Mittag-Leffler**, **Barnes**, **Hardy**, and **Littlewood** as investigators, either of the general theory of oscillating and asymptotic series, or of the asymptotic expansions of particular classes of functions. Complete bibliographies are to be found in **Bromwich**'s 'Theory of Infinite Series,' and **Barnes**' 'Memoir on Integral Functions.'* The former work contains an excellent history of the subject.

It might be considered that, when the theories of **Poincaré** and **Borel** had been discussed with such vigour, there would be comparatively little room for further general developments of the subject. In this memoir, however, I propose to discuss an aspect of the theory which has hitherto remained unnoticed, and which promises to have many useful applications. In fact, I have already found it to be of importance in connection with the problem of expanding an arbitrary function in a series of inverse factorials, and it is highly probable that the theory can be employed with advantage in particular cases of the problem of expanding an arbitrary function in a series of normal functions. A simpler application is that of determining an upper limit to the number of terms that should be taken in a given asymptotic series in order that, for a given value of the variable, the difference between the asymptotic expansion and the analytic function represented by the expansion should be as small as possible.

It is convenient here to specify precisely the meaning of certain expressions used in the sequel:—

(i) If x be a real variable, the statement $x > a$ means that some positive number Δ (independent of x) exists, such that $x \geqq a+\Delta$. Thus it might be convenient to take $\Delta = 10^{-100}$.

(ii) A function $f(x)$, of a complex variable x, is said to be analytic in the region $|x| < a$ when the function has no singularities in the interior of the region, although it may have singularities on the boundary $|x| = a$.

We say that the function is analytic in the region $|x| \leqq a$ when some positive number Δ exists such that the function is analytic in the region $|x| < a+\Delta$.

(iii) A function $f(x)$ is said to be analytic in the sector $\alpha \leqq \arg x \leqq \beta$ when, if x_0 be any singularity of $f(x)$ in the finite part of the plane, x_0 is not within the sector, and the distance of x_0 from the boundary of the sector is at least equal to Δ. The statement *does not mean* that $f(x)$ is "regular about the point $x = \infty$," *i.e.*, that $f(x)$ can be expanded in a convergent series of negative integer powers of x if $|x|$ be sufficiently large.

(iv) When a region is specified by means of two inequalities (*e.g.*, $|\arg x| < \frac{1}{2}\pi$, $|x| < 1$) we mean the region in which both the inequalities are satisfied, unless an explicit statement is made to the contrary.

(v) It is necessary to make use of the ideas† of "Orders of Infinity" so frequently that attention is rarely directed to their use. Thus, if we have $|x_n| < \rho^n$ where ρ is finite, and n is any integer, no matter how large, we should, if necessary, say at once that $|x_n| < \mathrm{K} \, . \, n!$ where K is finite.

Before proceeding, it is desirable to summarize the chief points of the theories of **Poincaré** and **Borel** :—

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* 'Phil. Trans.,' A, vol. 199, pp. 411–500, 1900.

† **Hardy**, "Orders of Infinity" ('Cambridge Tracts in Mathematics,' No. 12).

(i) A function $f(x)$ is said to possess an asymptotic expansion (in the sense of POINCARÉ) for large values of $|x|$, and for a certain range of values of arg x, if, for such values of arg x, the function can be expressed in the form

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n} + \mathrm{R}_n,$$

where $|\mathrm{R}_n| \leq \mathrm{J}_n |x|^{-n-1}$ when $|x| > \gamma$.

The quantity γ is finite, n is any assigned finite integer, and J_n is a finite quantity depending on γ and n, but not on $|x|$.

(ii) BOREL's theory is an attempt to associate with the series

$$a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots,$$

a quantity, S, which shall be equal to the sum of the series if the series happens to be convergent, and which shall have a definite meaning if the series be divergent.

Putting

$$a_0 + \frac{a_1 t}{1!} + \frac{a_2 t^2}{2!} + \ldots = \phi(t),$$

BOREL defines S by the equation

$$\mathrm{S} = \int_0^\infty e^{-t} \phi(t/x)\, dt\,;$$

when the integral converges, the original series is said to be "summable."

This theory requires some knowledge of the singularities of the function ϕ. Such a requirement is a defect of the theory, so far as many applications to asymptotic expansions are concerned, for it will often happen that, when we are given a function $f(x)$, we can obtain an asymptotic expansion of POINCARÉ's type with some knowledge of the values (or the upper limits of the values) of $a_0, a_1, \ldots, a_n, \mathrm{R}_n$; we may be able thence to deduce the radius of convergence* of the series for the function $\phi(t)$, and yet have no knowledge of the singularities of $\phi(t)$ outside the circle of convergence of the series for $\phi(t)$. By this lack of knowledge BOREL's theory is robbed of a great deal of its usefulness.†

Some severe strictures‡ have been passed by MITTAG-LEFFLER upon BOREL's theory for other reasons.

After these preliminary statements, we summarize the objects of this paper. In Part I. we define certain quantities called *characteristics*, which bear much the same relation to an asymptotic series as the radius of convergence bears to a convergent series. The definition is a natural consequence of an attempt to impart more precision to POINCARÉ's theory by making use of some of BOREL's ideas, but without making use of any of the properties of BOREL's associated function, $\phi(t)$, so far as the singularities of this function outside its circle of convergence are concerned. After defining the characteristics of a series, we proceed to prove a number of simple theorems concerning the characteristics of series derived in various manners from a series with given characteristics.

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* Which may be finite, as in the case when $a_n = (-)^n \cdot n!$.

† See, *e.g.*, the assumptions made in § 104 of BROMWICH'S 'Theory of Infinite Series.'

‡ At the Fourth International Congress of Mathematicians. See the 'Bulletin of the American Mathematical Society,' vol. xiv. (ser. 2), p. 485.

So far the analysis would not appear to have any practical importance. To justify its existence we investigate, in Part II., the circumstances in which an analytic function, known to possess an asymptotic expansion with assigned characteristics, *is determined uniquely by its asymptotic expansion.*[*] It is then possible to determine circumstances in which an asymptotic expansion, with given characteristics, is "summable" by the method of BOREL.

Finally we investigate the characteristics of the asymptotic expansion of a function derived from the "logarithmic-integral" function, as a simple example. Further examples, namely, of the gamma function and of MITTAG-LEFFLER's function, $E_\alpha(x)$, are contained in another paper by the writer.[†]

It should be mentioned here that the theory described in this memoir does not cover the investigation of those functions for which BOREL's associated function, $\phi(t)$, has a sequence of singularities at the points $t_1, t_2, \ldots$, such that $\underset{n=\infty}{\underline{Lt}} \arg t_n = 0$; such an associated function would be, *e.g.*, $\cot \pi(t+i)$, which has singularities at the points $t = -i+n$ (n any integer). It seems, however, that none of the ordinary functions of analysis possess associated functions of this nature.

PART I.—THE CHARACTERISTICS OF ASYMPTOTIC SERIES.

1. The type of function,[‡] $f(x)$, which we shall discuss, is subject to the condition that, when $|x| > \gamma$, $\alpha \leq \arg x \leq \beta$ (where α, β, γ are given finite quantities), it can be expressed in the form

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n} + R_n, \quad \ldots \ldots \ldots \quad (1)$$

where

$$|a_n| \leq A \,.\, \Gamma(kn+1) \,.\, \rho^n, \quad \ldots \ldots \ldots \quad (1\text{A})$$

$$|R_n x^{n+1}| \leq B \,.\, \Gamma(ln+1) \,.\, \sigma^n. \quad \ldots \ldots \ldots \quad (1\text{B})$$

The number n is any integer, and the quantities A, B, k, l, ρ, σ are independent of n and $|x|$.

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[*] It is known that an asymptotic expansion of POINCARÉ's type does not determine an analytic function uniquely; thus $(1+x)^{-1}$ and $(1+x)^{-1}+e^{-x}$ have the same asymptotic expansion, viz., $1-x^{-1}+x^{-2}-\ldots$, when $|x|$ is large and $|\arg x| < \frac{1}{2}\pi$.

[†] To be published shortly in the 'Quarterly Journal of Mathematics.'

[‡] Throughout Part I. of the paper the functions are not restricted so as to be analytic.

The novelty introduced in these assumptions lies in the fact that we consider a_n and R_n (or, more precisely, the moduli of these quantities) as functions of n for all values of n, no matter how large.

It will be convenient to give names to the quantities A, B, k, l, ρ, σ; we shall call k, l, ρ, σ *characteristics* of the series (1); k will be called a *grade*, l an *outer grade*, ρ a *radius*, σ an *outer radius* of the expansion. The quantities A and B will be called *constants* of the expansion.

If an inequality of the form (1A) can exist when $k \geqq k_0$, but not when $k < k_0$, we might, for greater precision, call k_0 the *principal grade* of the series; and the smallest possible value of ρ associated with k_0 might, in like manner, be called the *principal radius;* similarly we could define the *principal outer grade* and the *principal outer radius.* But it is found that this additional precision is unnecessary, and we will accordingly prove all our propositions for any possible characteristics; and, further, it may be stated that the characteristics, determined for particular series (such as the asymptotic expansion of the gamma function) by the ordinary processes of analysis, are of such a magnitude that, although they may not be principal characteristics, we could not expect to obtain any additional knowledge of the functions investigated by knowing the actual values of the principal characteristics.

Examples:—

(i) Let

$$\log a_n = kn \log n + cn(1+\epsilon_n),$$

where $|\epsilon_n| \to 0$ as $n \to \infty$, and k is real and positive.

The *principal grade* is k.

The *principal radius* is (by the asymptotic expansion of the gamma function) $|\exp\{c-k(\log k-1)\}|$, provided that the upper limit of $R(cn\epsilon_n)$, as $n \to \infty$, is not $+\infty$.

If the upper limit of $R(cn\epsilon_n)$ is $+\infty$, the quantity $|\exp\{c-k(\log k-1)\}|+\delta$ is a *possible radius*, where δ is an arbitrary positive quantity, but not zero.

(ii) Let

$$\log a_n = kn \log n + cn\{\sqrt{(\log n)}+\epsilon_n\},$$

where k and c are real and positive.

The quantity $k+\delta_1$ is a *possible grade;* and δ_2 is a *possible radius;* δ_1 and δ_2 being arbitrary positive quantities, but not zero.

(iii) The reader may prove that, if $f(x)$ be defined by (1), then $\int_x^\infty \{f(x)-a_0-a_1x^{-1}\}\,dx$ has the same characteristics as $f(x)$ when $k \leqq 1$ and $l \leqq 1$, where the path of integration is the straight line which, when produced backwards, passes through the origin.

Theorem I.—If k, l, ρ, σ are possible characteristics of an expansion, then, if we regard l and σ as being given, we may always assume that the quantities k and ρ are such that

$$\text{(i) } k \leqq l; \qquad \text{(ii) if } k = l,\ \rho \leqq \sigma.$$

For from equation (1) we have

$$a_{n+1} = (R_n - R_{n+1})\,x^{n+1},$$

so that

$$|a_{n+1}| \leqq \{|R_n| + |R_{n+1}|\} |x|^{n+1},$$

i.e.,

$$|a_{n+1}| \leqq B\sigma^n \{\Gamma(ln+1) + \sigma |x|^{-1} \Gamma(ln+l+1)\}.$$

Now, for all values of n, $\Gamma(ln+1) < K\Gamma(ln+l+1)$, where K may be taken to be a finite number independent of n.

Therefore, since $|x| > \gamma$, assuming that γ is not zero, we have

$$|a_{n+1}| < B\sigma^n \Gamma(ln+l+1) \{K + \sigma\gamma^{-1}\}.$$

Comparing this equation with (1A), we see that l is a possible grade; in other words, if we are given a number l as an outer grade, and a number k, greater than l, as a grade, we may take l as a new possible grade.

We also see that if $k = l$, and we are given σ as an outer radius, and a number ρ greater than σ as a radius, we may take σ as a new possible radius.

2. Let us now consider the product of two functions $f_1(x)$, $f_2(x)$ whose expansions (valid over a common range of values of arg x) are

$$f_1(x) = a_0 + \frac{a_1}{x} + \ldots + \frac{a_n}{x^n} + R_n,$$

$$f_2(x) = a'_0 + \frac{a'_1}{x} + \ldots + \frac{a'_n}{x^n} + R'_n.$$

Let possible constants and characteristics of these expansions be A_1, B_1, k_1, l_1, ρ_1, σ_1; A_2, B_2, k_2, l_2, ρ_2, σ_2 respectively.

We shall show that, for the range of values of arg x for which both these expansions are valid, the product $f_1(x) f_2(x)$ can be represented by an asymptotic series of which possible characteristics are k_0, l_0, ρ_0, σ_0; where k_0, l_0, ρ_0, σ_0 denote the greater of the numbers k_1, k_2; l_1, l_2; ρ_1, ρ_2; σ_1, σ_2 respectively.

By direct multiplication

$$f_1(x) f_2(x) = b_0 + \frac{b_1}{x} + \ldots + \frac{b_n}{x^n} + S_n,$$

where

$$b_n = a_0 a'_n + a_1 a'_{n-1} + \ldots + a_n a'_0,$$

$$S_n = a_0 R'_n + \frac{a_1}{x} R'_{n-1} + \frac{a_2}{x^2} R'_{n-2} + \ldots + \frac{a_n}{x^n} R'_0 + R_n (a'_0 + R'_0),$$

so that

$$|b_n| \leqq A_1 A_2 \sum_{r=0}^{n} \Gamma(rk_1+1) \Gamma\{(n-r)k_2+1\} \rho_1^r \rho_2^{n-r}.$$

Now, when* $\xi > 0$, $\Gamma(1+\xi)$ is positive and decreases as ξ increases till $\xi = 0{\cdot}461\ldots (= \xi_0$, say); when $\xi > \xi_0$, $\Gamma(1+\xi)$ increases with ξ; and when $0 < \xi < 1$, $\Gamma(1+\xi) < 1$.

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* DE MORGAN'S 'Differential and Integral Calculus' (1842), p. 590.

Also $\Gamma(1+\xi_0) = 0{\cdot}8856\ldots = \eta_0^{-1}$, say.

So that if

$$\xi > \xi' > 0, \quad \Gamma(1+\xi') < \eta_0 \Gamma(1+\xi);$$

and if, further,

$$\xi \geq 1, \qquad \Gamma(1+\xi') < \Gamma(1+\xi).$$

Returning to the definition of b_n, we get from these results

$$\begin{aligned} |b_n| &\leq A_1A_2 \sum_{r=0}^{n} \Gamma(rk_1+1)\,\Gamma\{(n-r)k_2+1\}\,\rho_1{}^r\rho_2{}^{n-r}, \\ &\leq A_1A_2\rho_0{}^n \left[\Gamma(nk_2+1)+\Gamma(nk_1+1)+\sum_{r=1}^{n-1}\Gamma(rk_1+1)\,\Gamma\{(n-r)k_2+1\}\right], \\ &\leq A_1A_2\rho_0{}^n \left[(1+\eta_1)\,\Gamma(nk_0+1)+\eta_1\sum_{r=1}^{n-1}\Gamma(rk_0+1)\,\Gamma\{(n-r)k_0+1\}\right], \end{aligned}$$

where

$$\eta_1 = \eta_0 \text{ if } k_0 < 1 \text{ and } k_1 \neq k_2,$$

and

$$\eta_1 = 1 \text{ if either or both of the conditions } k_0 \geq 1,\ k_1 = k_2 \text{ are satisfied.}$$

Now consider $F(\xi) = \Gamma(\xi k_0+1)\,\Gamma\{(n-\xi)k_0+1\}$, *qua* function of a continuous real variable ξ. If ψ denote the logarithmic derivate of the gamma function, we have

$$\frac{d}{d\xi}F(\xi) = k_0\,F(\xi)[\psi(\xi k_0+1)-\psi\{(n-\xi)k_0+1\}].$$

But, since $\dfrac{d}{d\xi}\psi(\xi) = \sum_{m=0}^{\infty}\dfrac{1}{(\xi+m)^2}$, we see that $\psi(\xi)$ increases with ξ when $\xi > 0$. Hence $\dfrac{d}{d\xi}F(\xi)$ is negative when $\xi \leq n-\xi$; and therefore, in the summation

$$\sum_{r=1}^{n-1}\Gamma(rk_0+1)\,\Gamma\{(n-r)k_0+1\},$$

the terms decrease until the middle term (or terms) and then increase, terms equidistant from the beginning and end of the summation being equal.

Two distinct cases now come under our consideration, (I.) when $k_0 \geq 1$, (II.) when $k_0 < 1$.

In the first case, using the equality

$$\Gamma(k_0+1)\,\Gamma\{(n-1)k_0+1\} = \Gamma(nk_0+1)\int_0^1 (1-z^{1/k_0})^{(n-1)k_0}dz,$$

we have, from the result just proved,

$$\begin{aligned} \sum_{r=1}^{n-1}\Gamma(rk_0+1)\,\Gamma\{(n-r)k_0+1\} &\leq (n-1)\,\Gamma(k_0+1)\,\Gamma\{(n-1)k_0+1\}, \\ &\leq (n-1)\,\Gamma(nk_0+1)\int_0^1(1-z^{1/k_0})^{(n-1)k_0}dz, \\ &\leq (n-1)\,\Gamma(nk_0+1)\int_0^1(1-z)^{(n-1)k_0}dz, \qquad \left[\begin{matrix}\text{since}\\ 1-z^{1/k_0} \leq 1-z\end{matrix}\right] \\ &\leq (n-1)\,\Gamma(nk_0+1)\,\{(n-1)k_0+1\}^{-1}, \\ &< k_0^{-1}\,\Gamma(nk_0+1). \end{aligned}$$

From this we deduce* that

$$|b_n| < A_1 A_2 \rho_0^n (2+k_0^{-1}) \Gamma(nk_0+1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the second case, let m be the greatest integer such that $mk_0 < 1$.

When $n \leqq 2m+1$, we have, as in the first case,

$$\begin{aligned}\sum_{r=1}^{n-1} \Gamma(rk_0+1)\, \Gamma\{(n-r)k_0+1\} &\leqq (n-1)\, \Gamma(k_0+1)\, \Gamma\{(n-1)k_0+1\}, \\ &\leqq (n-1)\, \Gamma(nk_0+1) \int_0^1 (1-z^{1/k_0})^{(n-1)k_0} dz, \\ &\leqq (n-1)\, \Gamma(nk_0+1), \qquad \left[\text{since the integrand is less than or equal to unity}\right] \\ &\leqq 2m\, \Gamma(nk_0+1),\end{aligned}$$

so that, when* $n \leqq 2m+1$, we deduce from the inequality on the preceding page :

$$|b_n| < A_1 A_2 \rho_0^n [1+\eta_1+2\eta_1 k_0^{-1}]\, \Gamma(nk_0+1).$$

Lastly, when $n \geqq 2m+2$, we have

$$\begin{aligned}&\sum_{r=1}^{n-1} \Gamma(rk_0+1)\, \Gamma\{(n-r)k_0+1\} \\ &\quad = 2\sum_{r=1}^{m} \Gamma(rk_0+1)\, \Gamma\{(n-r)k_0+1\} + \sum_{r=m+1}^{n-1-m} \Gamma(rk_0+1)\, \Gamma\{(n-r)k_0+1\}, \\ &\quad \leqq 2\sum_{r=1}^{m} \Gamma\{(n-r)k_0+1\} + (n-1-2m)\, \Gamma\{(m+1)k_0+1\}\, \Gamma\{(n-m-1)k_0+1\},\end{aligned}$$

since $\Gamma(rk_0+1) < 1$ when $1 \leqq r \leqq m$, and the largest terms of the summation $\sum_{m+1}^{n-1-m}$ are the first and last.

But, when $1 \leqq r \leqq m$, $\Gamma\{(n-r)k_0+1\} \leqq \Gamma(nk_0+1)$; and hence

$$\begin{aligned}\sum_{r=1}^{n-1} \Gamma(rk_0+1)\, \Gamma\{(n-r)k_0+1\} &\leqq 2m\Gamma(nk_0+1) \\ &\qquad + (n-1-2m)\, \Gamma(nk_0+1) \int_0^1 \{1-z^{1/(m+1)k_0}\}^{(n-m-1)k_0} dz, \\ &\leqq 2m\Gamma(nk_0+1) + (n-1-2m)\, \Gamma(nk_0+1) \int_0^1 (1-z)^{(n-m-1)k_0} dz, \\ &\qquad \left[\text{since } 1-z^{1/(m+1)k_0} \leqq 1-z\right] \\ &\leqq \{2m+k_0^{-1}\}\, \Gamma(nk_0+1),\end{aligned}$$

and consequently, in the second case, for all values of n, we have

$$|b_n| < A_1 A_2 \rho_0^n \{1+\eta_1+\eta_1(2m+k_0^{-1})\}\, \Gamma(nk_0+1) \quad . \quad . \quad . \quad . \quad . \quad (2\text{A})$$

We see at once that this formula covers the first case also (m being then zero)

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* This is only proved when $n \geqq 1$; but it is obviously true when $n = 0$.

We have thus proved that $f_1(x)f_2(x)$ *possesses an asymptotic expansion of which* k_0 *is a grade and* ρ_0 *is a radius, and a possible constant is*

$$A_1A_2\{1+\eta_1(2m+1+k_0^{-1})\},$$

where m is the greatest integer such that $mk_0 < 1$, and η_1 is defined as above.

We now wish to determine an outer grade and an outer radius.

We have seen that

$$S_n = a_0R'_n+\frac{a_1}{x}R'_{n-1}+\ldots+\frac{a_n}{x^n}R'_n+R_n(a'_0+R'_0),$$

so that*

$$|S_nx^{n+1}| < A_1B_2\sum_{r=0}^{n}\Gamma(rk_1+1)\Gamma\{(n-r)l_2+1\}\rho_1^r\sigma_2^{n-r}+B_1\Gamma(nl_1+1)\sigma_1^n(A_2+B_2\gamma^{-1}).$$

Let us suppose that $l_2 \geqq l_1$, so that $l_2 \geqq k_1$; then, since $\rho_1 \leqq \sigma_0$, $\sigma_2 \leqq \sigma_0$, we get, as in the previous work,

$$\begin{aligned}|S_nx^{n+1}| < {} & A_1B_2\{1+\eta'_1(2m'+1+l_0^{-1})\}\Gamma(nl_0+1)\sigma_0^n \\ & +A_2B_1\eta''_1\Gamma(nl_0+1)\sigma_0^n+B_1B_2\eta''_1\Gamma(nl_0+1)\gamma^{-1}\sigma_0^n,\end{aligned}$$

where m' is the greatest integer, such that $m'l_0 < 1$,

$$\eta'_1 = \eta_0 \quad \text{if} \quad l_0 > k_1 \quad \text{and} \quad l_0 < 1 \quad (\text{otherwise } \eta'_1 = 1),$$

$$\eta''_1 = \eta_0 \quad \text{if} \quad l_0 > l_1 \quad \text{and} \quad l_0 < 1 \quad (\text{otherwise } \eta''_1 = 1).$$

We consequently get always

$$|S_nx^{n+1}| < [(A_1B_2+A_2B_1)\{1+\eta_2(2m'+1+l_0^{-1})\}+B_1B_2\eta_2\gamma^{-1}]\sigma_0^n\Gamma(nl_0+1),$$

where $\eta_2 = \eta_0$ if $l_0 < 1$, $\eta_2 = 1$ if $l_0 \geqq 1$.

That is to say, l_0 *is an outer grade,* σ_0 *an outer radius, and*

$$(A_1B_2+A_2B_1)\{1+\eta_2(2m'+1+l_0^{-1})\}+B_1B_2\eta_2\gamma^{-1}$$

is a possible outer constant of the asymptotic expansion of $f_1(x)f_2(x)$.

We have thus obtained the results stated at the beginning of the section.

We deduce that if two functions $f_1(x)$, $f_2(x)$ both have asymptotic expansions with k, l, ρ, σ as a set of characteristics, and possible constants being A_1, B_1; A_2, B_2 respectively, then the product $f_1(x)f_2(x)$ has an asymptotic expansion with the same characteristics, the constants being A_0, B_0, where

$$A_0 = A_1A_2(2m+2+k^{-1}), \quad B_0 = (A_1B_2+A_2B_1)\{1+\eta_2(2p+1+l^{-1})\}+B_1B_2\gamma^{-1}, \quad (3\text{A, B})$$

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* The quantity γ is defined in connection with equation (1).

where m, p are the greatest integers, such that

$$mk < 1, \qquad pl < 1,$$

and

$$\eta_2 = \eta_0 \text{ if } l < 1, \qquad \eta_2 = 1 \text{ if } l \geqq 1.$$

Also, by induction, we deduce that if $f(x)$ have an asymptotic expansion with k, l, ρ, σ as characteristics, $\{f(x)\}^r$ has an asymptotic expansion with the same characteristics, r being any finite positive integer.

Let us, however, examine the series for $\{f(x)\}^r$ in greater detail when r is a positive integer.

Changing the notation slightly, let A, B, k, l, ρ, σ be possible constants and characteristics of $f(x)$; we may therefore take A_r, B_r, k, l, ρ, σ as constants and characteristics of $\{f(x)\}^r$, where

$$A_r = \lambda A_1 A_{r-1}, \qquad B_r = (A_1 B_{r-1} + A_{r-1} B_1)\lambda' + B_1 B_{r-1} \gamma^{-1}.$$

This follows from (3A) and (3B) combined with the identity

$$\{f(x)\}^r \equiv f(x)\{f(x)\}^{r-1};$$

we have written A_1, B_1 for A, B, and we have also written λ, λ' for the quantities $2m+2+k^{-1}$, $1+\eta_2(2p+1+l^{-1})$ of (3A) and (3B).

We see at once that

$$A_r = \lambda^{r-1}A^r, \qquad B_r = (A\lambda' + B\gamma^{-1})B_{r-1} + A^{r-1}B\lambda^{r-2}\lambda'. \quad . \quad . \quad . \quad (4\text{A}, \text{B})$$

Dividing through (4B) by $(A\lambda'+B\gamma^{-1})^r$, putting $r = 2, 3, \ldots$ and adding the results, we get without difficulty

$$B_r = (A\lambda'+B\gamma^{-1})^{r-1}B + AB\lambda'(A\lambda'+B\gamma^{-1})^{r-2}\sum_{n=0}^{r-2}\left(\frac{A\lambda}{A\lambda'+B\gamma^{-1}}\right)^n.$$

From equation (4B) we see that we should get a rather larger value of B_r than is given by that equation if we put μ for both λ and λ', where

$$\mu = 1+\eta_2(2m+1+k^{-1}),$$

so that $\mu \geqq \lambda$, $\mu \geqq \lambda'$, since $m \geqq p$, $k^{-1} \geqq l^{-1}$.

The result is so much simpler that we shall do so; and the result we get is that B'_r is a possible outer constant of $\{f(x)\}^r$ where

$$B'_r = \gamma\{(A\mu+B\gamma^{-1})^r - (A\mu)^r\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

3. We shall now prove that, if $f(x)$ has an asymptotic expansion with k, l, ρ, σ as characteristics, then exp $\{f(x)\}$ possesses an asymptotic expansion with the same characteristics.

Let A, B be possible constants of $f(x)$; from the definition of $f(x)$ we have, with the usual notation,

$$\exp\{f(x)\} = \exp a_0 \,.\, \exp \mathrm{R}_0$$

$$= (\exp a_0)\left[1+\frac{\mathrm{R}_0}{1!}+\frac{\mathrm{R}_0^{\,2}}{2!}+\ldots+\frac{\mathrm{R}_0^{\,n}}{n!}+\frac{\mathrm{R}_0^{\,n+1}}{(n+1)!}+\ldots\right].$$

Now, by the work of the preceding section, we can expand $\mathrm{R}_0^{\,m}$ in an asymptotic series with characteristics k, l, ρ, σ; and the expansion will be of the form

$$\mathrm{R}_0^{\,m} = \frac{{}_mb_m}{x^m}+\frac{{}_mb_{m+1}}{x^{m+1}}+\ldots+\frac{{}_mb_n}{x^n}+{}_m\mathrm{S}_n, \quad \ldots\ldots\ldots \quad (6)$$

where, by (4A) and (5),

$$|{}_mb_r| \leq \mathrm{A}^m\lambda^{m-1}\Gamma(kr+1)\,\rho^r \qquad (r \geq m)$$

$${}_mb_r = 0 \qquad (r < m)$$

and

$$|x^{n+1}\,.\,{}_m\mathrm{S}_n| \leq \gamma\left[(\mathrm{A}\mu+\mathrm{B}\gamma^{-1})^m-(\mathrm{A}\mu)^m\right]\Gamma(ln+1)\,\sigma^n.$$

We may consequently write

$$\exp f(x) = c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots+\frac{c_n}{x^n}+\mathrm{T}_n$$

where

$$c_n = (\exp a_0)\left[\frac{{}_1b_n}{1!}+\frac{{}_2b_n}{2!}+\ldots+\frac{{}_nb_n}{n!}\right], \qquad c_0 = \exp a_0,$$

and

$$\mathrm{T}_n = (\exp a_0)\left[\frac{{}_1\mathrm{S}_n}{1!}+\frac{{}_2\mathrm{S}_n}{2!}+\ldots+\frac{{}_n\mathrm{S}_n}{n!}+\frac{\mathrm{R}_0^{\,n+1}}{(n+1)!}+\frac{\mathrm{R}_0^{\,n+2}}{(n+2)!}+\ldots\right].$$

Consequently, if $n > 0$,

$$|c_n| \leq |\exp a_0|\left[\sum_{m=1}^{n}\frac{\mathrm{A}^m\lambda^{m-1}}{m!}\Gamma(kn+1)\,\rho^n\right],$$

i.e.,

$$|c_n| \leq \lambda^{-1}|\exp a_0|\{\exp(\mathrm{A}\lambda)-1\}\,\Gamma(kn+1)\,\rho^n, \quad \ldots\ldots \quad (7\mathrm{A})$$

and

$$|c_0| = |\exp a_0|.$$

Further,

$$|\mathrm{T}_n| \leq |\exp a_0|\left[|x|^{-n-1}\sum_{m=1}^{n}|{}_m\mathrm{S}_nx^{n+1}|+\frac{|\mathrm{R}_0|^{n+1}}{(n+1)!}+\frac{|\mathrm{R}_0|^{n+2}}{(n+2)!}+\ldots\right]$$

$$\leq |\exp a_0|\left[|x|^{-n-1}\sum_{m=1}^{n}\gamma\{(\mathrm{A}\mu+\mathrm{B}\gamma^{-1})^m-(\mathrm{A}\mu)^m\}\frac{\Gamma(ln+1)\,\sigma^n}{m!}\right.$$

$$\left.+\frac{\{\mathrm{B}|x|^{-1}\}^{n+1}}{(n+1)!}+\frac{\{\mathrm{B}|x|^{-1}\}^{n+2}}{(n+2)!}+\ldots\right],$$

so that

$$|T_n x^{n+1}\exp(-a_0)| \leq \left[\sum_{m=1}^{\infty}\gamma\frac{\{(A\mu+B\gamma^{-1})^m-(A\mu)^m\}}{m!}\right]\Gamma(ln+1)\,\sigma^n$$
$$+\frac{B^{n+1}}{(n+1)!}\left[1+\frac{B}{1!\,\gamma}+\frac{B^2}{2!\,\gamma^2}+\dots\right]$$
$$<\gamma\left[\exp(A\mu+B\gamma^{-1})-\exp(A\mu)\right]\Gamma(ln+1)\,\sigma^n+\frac{B^n}{(n+1)!}\exp\left(\frac{B}{\gamma}\right).$$

Now, by the asymptotic expansion of the gamma function, it follows that

$$\frac{B^{n+1}}{(n+1)!}<K\Gamma(ln+1)\,\sigma^n$$

for all integer values of n, where K is finite and independent of n, so that

$$|x^{n+1}T_n|<[\gamma\{\exp(A\mu+B\gamma^{-1})-\exp(A\mu)\}+K\exp(B\gamma^{-1})]\times|\exp a_0|\,.\,\Gamma(ln+1)\,\sigma^n. \quad (7\text{B})$$

Combining the equations (7A) and (7B), we see from (7) that exp $f(x)$ possesses an asymptotic expansion of which k, l, ρ, σ are characteristics.

4. We shall conclude this part of the paper by proving three general theorems.

Theorem II.—Let $f(x)$ *possess an asymptotic expansion with characteristics* k, l, ρ, σ, *and constants* A, B. *Then, if* $\Phi(\xi)$ *be a function of* ξ *which is regular** *inside a circle of radius greater than* $A\mu+B\gamma^{-1}$, *the centre of the circle being at* $\xi=a_0$ *(*μ *and* γ *having their usual significations), then* $\Phi\{f(x)\}$ *possesses an asymptotic expansion with characteristics* k, l, ρ, σ, *which is valid for the same range of values of* x *as the expansion of* $f(x)$.

Let $\Phi(a_0+\xi)$ be expansible in the series

$$\Phi(a_0+\xi)=g_0+g_1\xi+g_2\xi^2+\dots.$$

Then

$$\sum_{m=0}^{\infty}|g_m|(A\mu+B\gamma^{-1})^m=G,$$

and hence

$$|g_m|<G(A\mu+B\gamma^{-1})^{-m}$$

where G is finite and independent of m. [These statements are true if the less stringent condition be satisfied.]

Now, $f(x)=a_0+R_0$, so that

$$\Phi\{f(x)\}=g_0+g_1R_0+g_2R_0^{\,2}+\dots+g_nR_0^{\,n}+g_{n+1}R_0^{\,n+1}+\dots,$$

this series being convergent since $|R_0|<B\gamma^{-1}$.

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* This condition may be replaced by the slightly less stringent condition that the series for $\Phi(a_0+\xi)$ should be absolutely convergent when $|\xi|=A\mu+B\gamma^{-1}$.

Using the expansion (6) and keeping the notation of that equation, we find that

$$\Phi\{f(x)\} = c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots+\frac{c_n}{x^n}+\mathrm{T}_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where c_0, c_1, ..., T_n are now defined by the equations

$$c_0 = g_0, \qquad \mathrm{T}_0 = \sum_{m=1}^{\infty} g_m \mathrm{R}_0{}^m,$$

and, when $n > 0$,

$$c_n = \sum_{m=1}^{n} g_m \cdot {}_m b_n, \qquad \mathrm{T}_n = \sum_{m=1}^{n} g_m \cdot {}_m\mathrm{S}_n + \sum_{m=n+1}^{\infty} g_m \mathrm{R}_0{}^m.$$

Consequently, if $n > 0$,

$$|c_n| \leq \sum_{m=1}^{n} |g_n|\, \mathrm{A}^m \lambda^{m-1}\, \Gamma(kn+1)\, \rho^n$$

$$< \sum_{m=1}^{\infty} \lambda^{-1}\mathrm{G}\left\{\frac{\mathrm{A}\lambda}{\mathrm{A}\mu+\mathrm{B}\gamma^{-1}}\right\}^m \Gamma(kn+1)\,\rho^n,$$

i.e.,

$$|c_n| < \frac{\mathrm{AG}}{\mathrm{A}(\mu-\lambda)+\mathrm{B}\gamma^{-1}}\,\Gamma(kn+1)\,\rho^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\mathrm{A})$$

Also

$$|\mathrm{T}_n| \leq \sum_{m=1}^{n} |g_m|\,|{}_m\mathrm{S}_n| + \sum_{m=n+1}^{\infty} |g_m \mathrm{R}_0{}^m|,$$

$$\leq \sum_{m=1}^{n} |g_m|\,\gamma\,\{(\mathrm{A}\mu+\mathrm{B}\gamma^{-1})^m-(\mathrm{A}\mu)^m\}\,\Gamma(ln+1)\,\sigma^n |x|^{-n-1} + \sum_{m=n+1}^{\infty} |g_m|\,\{\mathrm{B}|x|^{-1}\}^m,$$

$$< \sum_{m=1}^{n} |g_m|\,\gamma\,(\mathrm{A}\mu+\mathrm{B}\gamma^{-1})^m\,\Gamma(ln+1)\,\sigma^n |x|^{-n-1} + \{\mathrm{B}|x|^{-1}\}^{n+1} \sum_{m=0}^{\infty} |g_{m+n+1}|\,\{\mathrm{B}|x|^{-1}\}^m,$$

$$< \gamma \mathrm{G}\,\Gamma(ln+1)\,\sigma^n |x|^{-n-1} + \mathrm{B}^{n+1}|x|^{-n-1} \sum_{m=0}^{\infty} \mathrm{G}\,\frac{(\mathrm{B}\gamma^{-1})^m}{(\mathrm{A}\mu+\mathrm{B}\gamma^{-1})^{m+n+1}},$$

i.e.,

$$|\mathrm{T}_n x^{n+1}| < \gamma\mathrm{G}\,\Gamma(ln+1)\,\sigma^n + \frac{\mathrm{GB}}{\mathrm{A}\mu}\left\{\frac{\mathrm{B}}{\mathrm{A}\mu+\mathrm{B}\gamma^{-1}}\right\}^n.$$

But, from the asymptotic expansion of the gamma function,

$$\left\{\frac{\mathrm{B}}{\mathrm{A}\mu+\mathrm{B}\gamma^{-1}}\right\}^n < \mathrm{K}'\,\Gamma(ln+1)\,\sigma^n,$$

where K' is finite and independent of n.

Hence

$$|\mathrm{T}_n x^{n+1}| < \{\gamma\mathrm{G}+\mathrm{GK}'\mathrm{BA}^{-1}\mu^{-1}\}\,\Gamma(ln+1)\,\sigma^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\mathrm{B})$$

Comparing (8A) and (8B) with (8), we see that $\Phi\{f(x)\}$ possesses an asymptotic expansion with characteristics, k, l, ρ, σ, valid for the range of values of x stated in the enunciation.

In particular, we notice that this theorem is true if Φ be an integral function. The result of the last section concerning $\exp\{f(x)\}$ is, of course, a particular case of the theorem.

5. *Theorem III.—Let* $f(x)$ *possess an asymptotic expansion with constants and characteristics* A, B, k, l, ρ, σ, *valid when* $|x| \geq \gamma$ *and for a certain range of values of* $\arg z$. *Then, if* $\Phi_1(a_0+\xi)$ *be a function of* ξ *which is regular when* $|\xi| \leq c$, *where* $c < \mathrm{A}\mu+\mathrm{B}\gamma^{-1}$ (μ *having its usual signification*), *then, when* $|x|$ *is greater than the two quantities* γ *and* B/c, $\Phi\{f(x)\}$ *possesses an asymptotic expansion, valid for the same range of values of* $\arg x$ *as the expansion for* $f(x)$, *with characteristics* k, l, ρ_0, σ_0, *where* ρ_0 *is the greater of the quantities*[*] ρ *and* $\rho\,.\,(\mathrm{A}\lambda/c)$, *while*

$$\sigma_0 = (\mathrm{A}\mu+\mathrm{B}\gamma_1^{-1})\,\sigma/c,$$

where γ_1 *is the larger of the quantities* γ *and* B/c.

Let the expansion of $\Phi_1(a_0+\xi)$, when $|\xi| \leq c$ be

$$\Phi_1(a_0+\xi) = h_0+h_1\xi+h_2\xi^2+\ldots;$$

since this series is absolutely convergent when $|\xi| = c$, we have

$$\sum_{m=0}^{\infty} |h_m|\,c^m < \mathrm{H}, \qquad |h_m| < \mathrm{H}c^{-m},$$

where H is finite and independent of m.

Since $f(x) = a_0+\mathrm{R}_0$, we may expand $\Phi_1\{f(x)\}$ into the series

$$\Phi_1\{f(x)\} = h_0+h_1\mathrm{R}_0+h_2\mathrm{R}_0^2+\ldots+h_n\mathrm{R}_0^n+h_{n+1}\mathrm{R}_0^{n+1}+\ldots,$$

provided $|\mathrm{R}_0| \leq c$.

Since $|\mathrm{R}_0| \leq \mathrm{B}|x|^{-1}$, the expansion will be valid when

$$|x| \geq \gamma \quad \text{and} \quad \mathrm{B}|x|^{-1} \leq c.$$

Substituting for R_0, R_0^2, ..., R_0^n as in Theorem II., we get

$$\Phi_1\{f(x)\} = d_0+\frac{d_1}{x}+\ldots+\frac{d_n}{x^n}+\mathrm{U}_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$d_0 = h_0, \qquad \mathrm{U}_0 = \sum_{m=1}^{\infty} h_m\mathrm{R}_0^m,$$

and when $n > 0$

$$d_n = \sum_{m=1}^{n} h_m\,.\,{}_mb_n, \qquad \mathrm{U}_n = \sum_{m=1}^{n} h_m\,.\,{}_m\mathrm{S}_n+\sum_{m=n+1}^{\infty} h_m\mathrm{R}_0^n.$$

The quantities ${}_mb_n$, ${}_m\mathrm{S}_n$ are the same as those which occur in equation (6).

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[*] If $\mathrm{A}\lambda = c$, $\rho_0 = \rho+\delta$, where δ is an arbitrary positive quantity as small as we please.

Consequently, if $n > 0$,

$$|d_n| \leq \sum_{m=1}^{n} |h_m| A^m \lambda^{m-1} \Gamma(kn+1) \rho^n$$

$$< \sum_{m=1}^{n} \lambda^{-1} H (A\lambda/c)^m \Gamma(kn+1) \rho^n. \qquad \text{(9A)}$$

If $A\lambda \neq c$, we have

$$\sum_{m=1}^{n} (A\lambda/c)^m = \frac{(A\lambda/c)^n - 1}{A\lambda/c - 1} \, . \, A\lambda/c < \{(A\lambda/c)^n + 1\} \frac{A\lambda}{|A\lambda - c|}.$$

From this result, combined with (9A), we see that, if $A\lambda \neq c$, we take ρ_0, the inner radius of $\Phi_1\{f(x)\}$, to be the smaller of the quantities ρ, $\rho A\lambda/c$.

If $A\lambda = c$, we have

$$|d_n| < n \, . \, \lambda^{-1} H \Gamma(kn+1) \rho^n,$$

and if $\delta > 0$, we have $n\rho^n < K_1 (\rho+\delta)^n$, where K_1 is a finite quantity depending on ρ and δ, but not on n. That is to say, if $A\lambda = c$, $\rho+\delta$ is a possible radius of $\Phi_1\{f(x)\}$ and k is a possible grade.

Further,

$$|U_n| \leq \sum_{m=1}^{n} \gamma |x|^{-n-1} [(A\mu + B\gamma^{-1})^m - (A\mu)^m] |h_m| \Gamma(ln+1) \sigma^n + \sum_{m=n+1}^{\infty} |h_m| |R_0|^m,$$

i.e.,

$$|U_n x^{n+1}| < \sum_{m=1}^{n} \gamma H \{(A\mu + B\gamma^{-1})/c\}^m \Gamma(ln+1) \sigma^n + \sum_{m=n+1}^{\infty} H \left\{\frac{B/c}{|x|}\right\}^m |x|^{n+1}$$

$$< \gamma H [\{(A\mu + B\gamma^{-1}) c^{-1}\}^n - 1] \Gamma(ln+1) \sigma^n [(A\mu + B\gamma^{-1}) c^{-1} - 1]^{-1}$$

$$+ \left(\frac{B}{c}\right)^{n+1} H \left[1 - \frac{B}{c|x|}\right]^{-1}.$$

Using the inequalities $B < c|x|$, $(A\mu + B\gamma^{-1}) c^{-1} > 1$, and $(Bc^{-1})^{n+1} < K_2 \Gamma(ln+1) \sigma^n$, (where K_2 is independent of n), we get a formula of the form

$$|U_n x^{n+1}| < K_3 \{(A\mu + B\gamma^{-1}) \sigma c^{-1}\}^n \Gamma(ln+1),$$

where K_3 is independent of n. That is to say, l and σ_0, defined as above, are a possible outer grade and a possible outer radius of $\Phi_1\{f(x)\}$.

6. *Theorem IV.—Suppose that for a certain range of values of* arg $(x+a)$, $f(x+a)$ *possesses an asymptotic expansion in negative powers of* $x+a$ *valid when* $|x+a| \geq \gamma$ *with constants and characteristics* A, B, k, l, ρ, σ. *Then for the same range of values of* arg $(x+a)$, $f(x+a)$ *possesses an asymptotic expansion in negative powers of* x *valid when* $|x| \geq \gamma + |a|$ *with characteristics* k, l, ρ_1, σ_1, *where*

$$\rho_1 = \rho + |a|,$$

and σ_1 *is the greater of the quantities* $\sigma\mu$ *and* $|a|+\rho+\rho|a|\mu'$ *where* μ, μ' *are the greatest possible values of* $|x(x+a)^{-1}|$, $|(x+a)^{-1}|$ *respectively for the values of* x *under consideration.*

It is easily shown that

$$\frac{1}{x+a} = \frac{1}{x} - \frac{a}{x^2} + \frac{a^2}{x^3} - \dots + (-)^{n-1}\frac{a^{n-1}}{x^n} + (-)^n \frac{a^n}{x^n(x+a)}.$$

Differentiating r times with respect to a, we get

$$\frac{(-)^r r!}{(x+a)^{r+1}} = \frac{(-)^r . r!}{x^{r+1}} + \frac{(-)^{r+1} a\,(r+1)!}{x^{r+2} . 1!} + \dots + \frac{(-)^{n-1} a^{n-r-1}(n-1)!}{x^n . (n-r-1)!} + (-)^n r!\, Y_{r+1} \quad (10)$$

where

$$|x^n . r!\, Y_{r+1}| = \left|\frac{d^r}{da^r}\left(\frac{a^n}{x+a}\right)\right|.$$

But, by LEIBNIZ' theorem,

$$\frac{d^r}{da^r}\left(\frac{a^n}{x+a}\right) = \frac{n!\, a^{n-r}}{(x+a) . (n-r)!}\left[1 - {}_rC_1 \frac{a . 1}{(n-r+1)(x+a)} + {}_rC_2 \frac{a^2 . 1 . 2 .}{(n-r+1)(n-r+2)(x+a)^2} - \dots \right.$$
$$\left. + (-)^r {}_rC_r \frac{a^r . 1 . 2 . \dots r}{(n-r+1)(n-r+2)\dots(n)(x+a)^r}\right],$$

the quantities ${}_rC_1$, ${}_rC_2$, ... being the binomial coefficients.

Consequently

$$\left|\frac{d^r}{da^r}\left(\frac{a^n}{x+a}\right)\right| < \frac{n!\,|a|^{n-r}}{|x+a| . (n-r)!}\left[1 + {}_rC_1\left|\frac{a}{x+a}\right| + {}_rC_2\left|\frac{a}{x+a}\right|^2 + \dots\right]$$
$$< \frac{n!\, |a|^{n-r}}{|x+a| . (n-r)!}\left[1 + \left|\frac{a}{x+a}\right|\right]^r,$$

so that

$$|x^n . r!\, Y_{r+1}| < \frac{n!\,|a|^{n-r}}{|x+a| . (n-r)!}\left[1 + \frac{|a|}{|x+a|}\right]^r.$$

Now $f(x+a)$ possesses an expansion of the form

$$f(x+a) = a_0 + \frac{a_1}{x+a} + \dots + \frac{a_n}{(x+a)^n} + R_n,$$

where $|a_n| \leq A\rho^n\Gamma(kn+1)$, $|R_n(x+a)^{n+1}| \leq B\sigma^n\Gamma(ln+1)$.

Substituting for the negative powers of $x+a$ from (10), we get

$$f(x+a) = b_0 + \frac{b_1}{x} + \dots + \frac{b_n}{x^n} + S_n,$$

where

$$b_n = a_n - {}_{n-1}C_1 . a . a_{n-1} + {}_{n-1}C_2 . a^2 . a_{n-2} - \dots$$

and

$$\mathrm{S}_n = \mathrm{R}_n+(-)^n a_1\mathrm{Y}_1+(-)^{n-1}a_2\mathrm{Y}_2+\ \ldots\ -a_n\mathrm{Y}_n,$$

so that

$$\begin{aligned}|b_n| &\leq |a_n|+{}_{n-1}\mathrm{C}_1|a|\,|a_{n-1}|+{}_{n-1}\mathrm{C}_2|a|\,|a_{n-2}|+\ldots\\ &\leq \mathrm{A}\left[\Gamma(kn+1)\rho^n+\Gamma\{k(n-1)+1\}\rho^{n-1}|a|\,.\,{}_{n-1}\mathrm{C}_1+\Gamma\{k.(n-2)+1\}\rho^{n-2}|a|^2{}_{n-1}\mathrm{C}_2+\ldots\right].\end{aligned}$$

Now

$$\Gamma\{k(n-r)+1\}\leq\eta_1\Gamma(kn+1),$$

where

$$\eta_1^{-1} = \cdot 8856\ldots \text{ if } k<1, \qquad \eta_1 = 1 \text{ if } k\geq 1.$$

Consequently

$$|b_n|\leq \mathrm{A}\eta_1\Gamma(kn+1)\left[\rho^n+{}_n\mathrm{C}_1\rho^{n-1}|a|+{}_n\mathrm{C}_2\rho^{n-2}|a|^2+\ldots\right],$$

i.e.,

$$|b_n|\leq \mathrm{A}\eta_1\Gamma(kn+1)\{\rho+|a|\}^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10\mathrm{A})$$

Also

$$\begin{aligned}|\mathrm{S}_n| &\leq \mathrm{B}\Gamma(ln+1)\,\sigma^n|x+a|^{-n-1}+|a_1\mathrm{Y}_1|+|a_2\mathrm{Y}_2|+\ \ldots\ +|a_n\mathrm{Y}_n|\\ &\leq \frac{\mathrm{B}\Gamma(ln+1)\,\sigma^n}{|x+a|^{n+1}}+\sum_{r=0}^{n-1}\mathrm{A}\,\frac{n\,!\,|a|^{n-r}}{|x+a|(n-r)\,!}\left[1+\frac{|a|}{|x+a|}\right]^r\rho^{r+1}\,\frac{\Gamma(kr+k+1)}{r\,!\,|x|^n}.\end{aligned}$$

And for the values of r under consideration (since $k\leq l$),

$$\Gamma(kr+k+1)<\eta_1\Gamma(ln+1).$$

Therefore

$$\begin{aligned}|\mathrm{S}_n x^{n+1}| &\leq \Gamma(ln+1)\left[\left\{\frac{\sigma|x|}{|x+a|}\right\}^n\frac{\mathrm{B}|x|}{|x+a|}+\frac{\mathrm{A}\rho}{|x+a|}\left\{|a|+\rho+\frac{\rho|a|}{|x+a|}\right\}^n\right]\\ &\leq \Gamma(ln+1)\left[\mathrm{B}\mu\,.\,(\sigma\mu)^n+\mathrm{A}\rho\mu'\{|a|+\rho+\rho\mu'|a|\}^n\right]\end{aligned}$$

i.e.,

$$|\mathrm{S}_n x^{n+1}|\leq\Gamma(ln+1)\,.\,(\mathrm{B}\mu+\mathrm{A}\rho\mu')\,\sigma_1^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10\mathrm{B})$$

From (10A) and (10B) we see that k, l, ρ_1, σ_1 are characteristics of the expansion of $f(x+a)$ in descending powers of x.

We have now proved all the theorems which seem to be of importance concerning asymptotic series in general. We proceed to discuss properties of analytic functions of which asymptotic expansions are given.

PART II.—ANALYTIC FUNCTIONS DEFINED BY ASYMPTOTIC SERIES.

7. We first propose to consider the question of the uniqueness of an analytic function which is defined by means of an asymptotic expansion possessing given characteristics.

The discussion will be based on the result of the following important lemma, of which we shall give a proof before proceeding further :—

Lemma.—Let $f(x)$ be a function of x which is analytic in the sector defined by*

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* See (iii.) on p. 280.

the inequality $|\arg x| \leq \frac{1}{2}\pi+\lambda$ *where* $\lambda > 0$; *and let the region in which* $f(x)$ *is analytic be extended to all points (outside the sector) whose distance from the boundary of the sector does not exceed* 2Δ, *where* $\Delta > 0$.

Let $f(x)$ *be such that, throughout the sector and this extended region adjacent to the sector, there exists an inequality of the form*

$$|f(x)| < \mathrm{A} \exp\{-|x|\},$$

where A *is a constant independent of* x.

Then the function $f(x)$ *is identically zero.**

Let x_0 be any point within or on the boundary of the region $|\arg x| \leq \frac{1}{2}\pi+\lambda$.

Then

$$\frac{d^n f(x_0)}{dx_0{}^n} = \frac{n!}{2\pi i}\int \frac{f(t)}{(t-x_0)^{n+1}}\,dt,$$

round an appropriate contour.

Since any singularity of $f(x)$ is, at a distance, greater than or equal to 2Δ from x_0, we may take the contour to be a circle of radius Δ with x_0 as centre.

We then get, without difficulty,

$$|f^{(n)}(x_0)| \leq n!\,\Delta^{-n}\,.\,\mathrm{M}$$

where M is the greatest value of $|f(t)|$ on the contour, and $f^{(n)}$ denotes the nth differential coefficient of f.

But on the contour

$$\begin{aligned}|f(t)| &< \mathrm{A}\exp\{-|t|\}\\ &< \mathrm{A}\exp\{\Delta-|x_0|\} \qquad \text{since } |t| \geq |x_0|-\Delta.\end{aligned}$$

That is to say, if $|\arg x_0| \leq \frac{1}{2}\pi+\lambda$,

$$|f^{(n)}(x_0)| < n!\,e^{\Delta}\,.\,\Delta^{-n}\,.\,\mathrm{A}\exp\{-|x_0|\}. \qquad (11)$$

Let us denote the integral of a function taken along a line from the origin to infinity, inclined at an angle θ to the real axis, by the symbol $\int_{(\theta)}$.

Consider the function F defined by the equation

$$\mathrm{F}(y) = \int_{(0)} f(ty)\,e^{-\frac{1}{2}t}\,dt. \qquad (12)$$

The function F(y) is analytic† in the interior of the region given by $|\arg y| < \frac{1}{2}\pi+\lambda$ (but possibly it is not analytic on the boundary of the region).

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* The proof of the lemma is suggested by a paper by PHRAGMEN, 'Acta Mathematica,' vol. 28, pp. 351–368. His paper, however, deals with integral functions, whereas we know nothing at all about the behaviour of $f(x)$ outside a certain sector of the plane.

† The condition given by BROMWICH, 'Theory of Infinite Series,' p. 438, is satisfied by defining his function M(t) by the equation $\mathrm{M}(t) = \mathrm{A}e^{\Delta}\Delta^{-1}\,.\,te^{-\frac{1}{2}t}$.

We shall show that $\mathrm{F}(y)$ is a constant, independent of y.

If $\frac{1}{2}\pi+\lambda \geqq \arg y \geqq -\frac{3}{2}\lambda$, we have

$$\int_{(0)} f(ty)\, e^{-\frac{1}{2}t}\, dt = \int_{(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} f(ty)\, e^{-\frac{1}{2}t}\, dt\,;$$

for, by CAUCHY's theorem, the difference between these two integrals is $\int f(ty)\, e^{-\frac{1}{2}t}\, dt$ taken along the arc of an indefinitely great circle terminated by the lines $\arg t = 0$, $\arg t = -\frac{1}{2}\pi+\frac{1}{2}\lambda$; on this arc we have $|\arg(ty)| \leqq \frac{1}{2}\pi+\lambda$, $|\arg t| < \frac{1}{2}\pi$; it follows, without difficulty, that the integral along the arc is zero.

But $\int_{(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} f(ty)\, e^{-\frac{1}{2}t}\, dt$ is uniformly convergent over the interior of the region

$$\pi+\tfrac{1}{2}\lambda \geqq \arg y \geqq -\tfrac{3}{2}\lambda.$$

Consequently the analytic continuation of $\mathrm{F}(y)$ *over the interior of the region* $\pi+\frac{1}{2}\lambda \geqq \arg y \geqq -\frac{3}{2}\lambda$ *is given by the equation*

$$\mathrm{F}(y) = \int_{(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} f(ty)\, e^{-\frac{1}{2}t}\, dt. \quad \ldots\ldots\ldots \quad (12\text{A})$$

In like manner, the analytic continuation of $\mathrm{F}(y)$ over the interior of the region $-\pi-\frac{1}{2}\lambda \leqq \arg y \leqq \frac{3}{2}\lambda$ is given by the equation

$$\mathrm{F}(y) = \int_{(\frac{1}{2}\pi-\frac{1}{2}\lambda)} f(ty)\, e^{-\frac{1}{2}t}\, dt. \quad \ldots\ldots\ldots \quad (12\text{B})$$

Also $\mathrm{F}(0) = \int_0^\infty f(0)\, e^{-\frac{1}{2}t}\, dt = 2f(0)$; and we may show, by using equations (12A) and (12B), that $\underset{y=0}{\mathrm{Lt}}\, \mathrm{F}(y) = 2f(0)$ when y approaches the origin by any path which does not go outside the sector $\pi+\frac{1}{2}\lambda \geqq \arg y \geqq -\frac{3}{2}\lambda$, or which does not go outside the sector $-\pi-\frac{1}{2}\lambda \leqq \arg y \leqq \frac{3}{2}\lambda$.*

Now, *if we can show that* $\mathrm{F}(y)$ *is a uniform function of* y, the only possible singularities of $\mathrm{F}(y)$ will be at the points $y = 0$ and $y = \infty$; and the only possible branch point of $\mathrm{F}(y)$ in the finite part of the plane is at $y = 0$; accordingly, to prove the uniformity of $\mathrm{F}(y)$, it is sufficient to prove that, when y starts from any point in an assigned region of non-zero area (not including the origin) and describes a closed circuit round the origin, the initial and final values of $\mathrm{F}(y)$ are the same.†

Let y_0 be any quantity such that

$$|y_0| \geqq 1, \qquad -\pi-\tfrac{1}{2}\lambda < \arg y_0 < -\pi+\tfrac{1}{2}\lambda\,;$$

we proceed to prove that $\mathrm{F}(y_0 e^{2\pi i}) = \mathrm{F}(y_0)$.

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* The integrals (12A) and (12B) are uniformly convergent when y lies within or on the boundary of the respective sectors, by WEIERSTRASS' test for the uniform convergence of infinite integrals [BROMWICH, 'Theory of Infinite Series,' p. 434; we replace BROMWICH's function $\mathrm{M}(x)$ by $\mathrm{A}e^{-\frac{1}{2}t}$].

† If two analytic functions are equal at all points of a region of non-zero area they are the same branch of the same function; in the case under consideration the functions are $\mathrm{F}(y_0)$, $\mathrm{F}(y_0 e^{2\pi i})$.

Suppose that y starts from y_0 and describes a circle of radius $|y_0|$ and centre at the origin, ending at the point $y_0e^{2\pi i}$.

We have
$$\mathrm{F}(y_0) = \int_{(\frac{1}{2}\pi-\frac{1}{2}\lambda)} f(ty_0)\, e^{-\frac{1}{2}t}\, dt.$$

Making the point y move from y_0 to $y_0e^{\pi i}$ round the circle, we get
$$\mathrm{F}(y_0e^{\pi i}) = \int_{(\frac{1}{2}\pi-\frac{1}{2}\lambda)} f(ty_0e^{\pi i})\, e^{-\frac{1}{2}t}\, dt.$$

Now, when
$$\tfrac{1}{2}\pi-\tfrac{1}{2}\lambda \geqq \arg t \geqq -\tfrac{1}{2}\pi+\tfrac{1}{2}\lambda,$$
$$|\arg t| < \tfrac{1}{2}\pi, \qquad |\arg(ty_0e^{\pi i})| < \tfrac{1}{2}\pi+\lambda.$$

Consequently $\int f(ty_0)\, e^{-\frac{1}{2}t}\, dt$ taken round an arc of a circle of radius N terminated by the points $\mathrm{N}\exp\{\pm(\frac{1}{2}\pi-\frac{1}{2}\lambda)i\}$ tends to zero as $\mathrm{N}\rightarrow\infty$; *and therefore, by* CAUCHY'S *theorem, we may deform the path of integration and get*
$$\mathrm{F}(y_0e^{\pi i}) = \int_{(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} f(ty_0e^{\pi i})\, e^{-\frac{1}{2}t}\, dt.$$

Now make y move from $y_0e^{\pi i}$ to $y_0e^{2\pi i}$, and we get
$$\mathrm{F}(y_0e^{2\pi i}) = \int_{(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} f(ty_0e^{2\pi i})\, e^{-\frac{1}{2}t}\, dt.$$

Writing $te^{-2\pi i}$ for t, we get
$$\mathrm{F}(y_0e^{2\pi i}) = \int_{(\frac{3}{2}\pi+\frac{1}{2}\lambda)} f(ty_0)\, e^{-\frac{1}{2}t}\, dt.$$

Consequently
$$\mathrm{F}(y_0e^{2\pi i}) - \mathrm{F}(y_0) = \left[\int_{(\frac{3}{2}\pi+\frac{1}{2}\lambda)} - \int_{(\frac{1}{2}\pi-\frac{1}{2}\lambda)}\right] f(ty_0)\, e^{-\frac{1}{2}t}\, dt. \quad . \quad . \quad . \quad . \quad (13)$$

Now consider $\int f(ty_0)\, e^{-\frac{1}{2}t}\, dt$ taken round an arc of a circle of radius N terminated by the points
$$\mathrm{N}\exp\{(\tfrac{3}{2}\pi+\tfrac{1}{2}\lambda)i\}, \qquad \mathrm{N}\exp\{(\tfrac{1}{2}\pi-\tfrac{1}{2}\lambda)i\}.$$

On this arc, which we call Γ,
$$|\arg(ty_0)| < \tfrac{1}{2}\pi+\lambda.$$

Consequently on Γ,
$$|f(ty_0)| < \mathrm{A}\exp\{-|t|y_0\}.$$

Therefore
$$\left|\int_{\Gamma} f(ty_0)\, e^{-\frac{1}{2}t}\, dt\right| < \int_{\Gamma} \mathrm{A}\,[\exp\{-|t|\,.\,|y_0|\}\,|e^{-\frac{1}{2}t}|\, d|t|$$
$$< \mathrm{A}\int_{\Gamma} \exp\{-(|y_0|-\tfrac{1}{2})|t|\}\, d|t|$$
$$< \mathrm{A}\int_{\Gamma} \exp\{-\tfrac{1}{2}\mathrm{N}\}\, d|t|, \quad \text{since} \quad |y_0| \geqq 1.$$
$$< \mathrm{AN}(\pi+\lambda)\,.\,\exp\{-\tfrac{1}{2}\mathrm{N}\};$$

and this expression tends to zero as $\mathrm{N}\rightarrow\infty$.

Consequently, by CAUCHY'S *theorem, the right-hand side of equation* (13) *vanishes;* for the integrand has no singularities between Γ and the rays $\arg t = \frac{1}{2}\pi - \frac{1}{2}\lambda$, $\arg t = \frac{3}{2}\pi + \frac{1}{2}\lambda$.

In other words, $\mathrm{F}(y_0 e^{2\pi i}) = \mathrm{F}(y_0)$; that is to say, $\mathrm{F}(y)$ *is a uniform function of* y.

Furthermore, $|\mathrm{F}(y)|$ never exceeds a finite quantity independent of y.

For*

$$\begin{aligned} |\mathrm{F}(y)| &= \left| \int_{\pm(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} f(ty)\, e^{-\frac{1}{2}t}\, dt \right| \\ &< \mathrm{A} \int_{\pm(-\frac{1}{2}\pi+\frac{1}{2}\lambda)} |e^{-\frac{1}{2}t}| \,.\, |dt| \\ &< 2\mathrm{A} \operatorname{cosec} \tfrac{1}{2}\lambda. \end{aligned}$$

We have thus proved that $\mathrm{F}(y)$ is a uniform function of y whose modulus never exceeds a finite quantity independent of y, no matter how large or how small $|y|$ may be.

Therefore by LIOUVILLE'S *theorem* $\mathrm{F}(y)$ *is a pure constant.*

The proof that $f(y)$ is zero is now immediate.

For the equation

$$\frac{\partial}{\partial y} \int_0^\infty \phi(t, y)\, dt = \int_0^\infty \frac{\partial \phi}{\partial y}\, dt$$

is true provided the integral on the right converges uniformly and the integral on the left is convergent†.

Now

$$\begin{aligned} \left| \frac{\partial}{\partial y} \{ f(ty)\, e^{-\frac{1}{2}t} \} \right| &= | t e^{-\frac{1}{2}t} f'(ty) | \\ &< \mathrm{A} e^{\Delta} \Delta^{-1} t e^{-\frac{1}{2}t} \text{ by (11)} \quad \text{when} \quad t \geqq 0,\ y \geqq 0. \end{aligned}$$

Since $\int_0^\infty t e^{-\frac{1}{2}t}\, dt$ is convergent, we know that‡ $\int_0^\infty \frac{\partial}{\partial y} f(ty) \,.\, e^{-\frac{1}{2}t}\, dt$ converges uniformly when $y \geqq 0$.

Therefore
$$\frac{d}{dy} \mathrm{F}(y) = \int_0^\infty t f'(ty)\, e^{-\frac{1}{2}t}\, dt \quad \text{when} \quad y \geqq 0.$$

Put $y = 0$, and we get, since $\mathrm{F}(y)$ is a constant,

$$0 = \int_0^\infty t f'(0)\, e^{-\frac{1}{2}t}\, dt.$$

---

* If the imaginary part of y is positive or zero, we may take $0 \leqq \arg y \leqq \pi$, and the upper sign is to be taken in the ambiguity; if the imaginary part of y is negative or zero, we may take $0 \geqq \arg y \geqq -\pi$, and we take the lower sign; we have already discussed what happens when $y = 0$.

† BROMWICH, 'Theory of Infinite Series,' p. 437.

‡ *Ibid.*, p. 434.

In like manner, since $\int_0^\infty t^n e^{-\frac{1}{2}t}\,dt$ is convergent, we may show that, when n is any finite integer,

$$\frac{d^n}{dy^n}\mathrm{F}(y) = \int_0^\infty t^n f^{(n)}(ty)\,e^{-\frac{1}{2}t}\,dt \quad \text{when} \quad y \geqq 0,$$

and $f^{(n)}$ denotes the n^{th} differential coefficient of f.

Putting $y = 0$, we get

$$0 = f^{(n)}(0)\int_0^\infty t^n e^{-\frac{1}{2}t}\,dt, \quad i.e., \ f^{(n)}(0) = 0.$$

Therefore $f(y)$ is analytic when $|y| < 2\Delta$ and all the differential coefficients of $f(y)$ vanish when $y = 0$; that is to say, $f(y)$ is a pure constant, which we will call L.

Furthermore, by the definition of $f(y)$

$$\mathrm{L} < \mathrm{A}\exp(-|y|)$$

when $|\arg y| \leqq \frac{1}{2}\pi + \lambda$, for all values of $|y|$, no matter how large; since $\exp(-|y|) \rightarrow 0$ as $|y| \rightarrow \infty$, we infer that $\mathrm{L} = 0$.

We have thus proved the lemma, that

$$f(y) = 0.$$

8. We are now in a position to discuss the uniqueness of an analytic function possessing an asymptotic expansion with given characteristics for a certain range of values of the argument of the variable.

The theorem, stated precisely, is as follows:

Theorem V.—Let there be two functions $f_1(x)$, $f_2(x)$, *which are analytic in the region defined by the inequalities*

$$|x| \geqq \gamma, \qquad \alpha \leqq \arg x \leqq \beta;$$

and let them be such that in this region they possess the asymptotic expansions

$$f_1(x) = a_0 + \frac{a_1}{x} + \ldots + \frac{a_n}{x^n} + \mathrm{R}_n,$$

$$f_2(x) = a_0 + \frac{a_1}{x} + \ldots + \frac{a_n}{x^n} + \mathrm{S}_n,$$

where, for all values of n,

$$|a_n| < \mathrm{A}\Gamma(kn+1)\,\rho^n, \quad |\mathrm{R}_n| < \mathrm{B}\Gamma(ln+1)\,\sigma^n|x|^{-n-1}, \quad |\mathrm{S}_n| < \mathrm{B}\Gamma(ln+1)\,\sigma^n|x|^{-n-1}.$$

Then, if $\beta - \alpha > \pi l$,

$$f_1(x) \equiv f_2(x).$$

Let the region in which x is permitted to lie be called the region C.

Since

$$|f_1(x) - f_2(x)| < |\mathrm{R}_n| + |\mathrm{S}_n|,$$

we have

$$|f_1(x)-f_2(x)| < 2B\Gamma(ln+1)\,\sigma^n|x|^{-n-1}$$

for all values of n, provided x lie in the region C.

Now choose n to depend on x in such a way that

$$n \leq l^{-1}\{|x\sigma^{-1}|\}^{1/l} < n+1,$$

so that we may put

$$n = l^{-1}\{|x\sigma^{-1}|\}^{1/l}-\theta,$$

where $0 \leq \theta < 1$.

Now let γ' be the greater of the two quantities $(1+l)^l\,\sigma$ and γ; and let the region in which $|x| > \gamma'$, $\alpha \leq \arg x \leq \beta$ be called the region C′.

When x lies in the region C′, we have

$$ln = \{|x\sigma^{-1}|\}^{1/l}-l\theta > (1+l)-l\theta > 1.$$

But when $ln > 1$, by the asymptotic expansion of the gamma function,

$$\log\Gamma(ln+1) = (ln+\tfrac{1}{2})\log(ln+l\theta)-ln-l\theta+\tfrac{1}{2}\log 2\pi+J,$$

where J does not exceed a finite quantity independent of n. Consequently, when x lies in the region C′,

$$|f_1(x)-f_2(x)| < B_1\exp\left[(ln+\tfrac{1}{2})\log(ln+l\theta)-ln-l\theta-(n+1)\log|x\sigma^{-1}|\right],$$

where B_1 is a finite quantity independent of n and x.

Substituting for $ln+l\theta$ in terms of x, we get

$$|f_1(x)-f_2(x)| < B_1|x\sigma^{-1}|^{-1+1/(2l)}\exp\left[-|x\sigma^{-1}|^{1/l}\right].$$

Putting $f_1(x)-f_2(x) = f_3(x)$, we want to show that if $f_3(x)$ is analytic in the region C′ and subject to the inequality

$$|f_3(x)| < B_1|x\sigma^{-1}|^{-1+1/(2l)}\exp\left[-|x\sigma^{-1}|^{1/l}\right],$$

then $f_3(x) = 0$.

Let us put[*] $x = \sigma y^l$ and $f_3(x) = f_4(y)$.

Then $f_4(y)$ is analytic in the region, C″, in which

$$|y| > \sigma^{-1/l}(\gamma')^{1/l}, \qquad \alpha \leq l\arg y \leq \beta,$$

and $f_4(y)$ is subject to the inequality (when y lies in C″)

$$|f_4(y)| < B_1|y|^{-l+\frac{1}{2}}\exp\{-|y|\}.$$

If $l \geq \frac{1}{2}$, we see at once that in C″

$$|f_4(y)| < B_2\exp\{-|y|\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where B_2 is a finite constant depending on B_1 and γ'.

---

[*] $y = \infty$ is a singularity of this transformation; but see (iii.) on p. 280.

Also $|y|^{-l+\frac{1}{2}} \exp\{-\frac{1}{2}|y|\}$ decreases when $|y| > 1-2l$ if $l < \frac{1}{2}$.

Let γ_2 be the greater of the quantities $(\sigma^{-1}\gamma')^{1/l}$, $1-2l$.

Calling the region in which $|y| > \gamma_2$, $\alpha \leq l \arg y \leq \beta$, the region C_2, we see from (14) and this result that in the region C_2, whether l be greater than or less than $\frac{1}{2}$, we have an inequality of the form $|f_4(y)| < B_3 \exp\{-\frac{1}{2}|y|\}$, and y is analytic in the region C_2.

Now, define a quantity* λ such that $\frac{1}{2}\pi > \lambda > 0$, $(\beta-\alpha)/l \geq \pi+2\lambda$, and put

$$z = \tfrac{1}{2}y \exp\{-\tfrac{1}{2}(\alpha+\beta)i\} - (\tfrac{1}{2}\gamma_2+2\Delta)\sec\lambda,$$

where $\Delta > 0$; and let $f_4(y) \equiv f(z)$.

Then $f(z)$ is certainly analytic in the sector $|\arg z| \leq \frac{1}{2}\pi+\lambda$, and at all points at a distance not greater than 2Δ from the boundary of the sector; for when z lies in the region just specified, y certainly lies in the region C_2.

Also, in the region specified for z,

$$\begin{aligned} |f(z)| &< B_3 \exp\{-|z+(\tfrac{1}{2}\gamma_2+2\Delta)\sec\lambda|\} \\ &< B_3 \exp\{(\tfrac{1}{2}\gamma_2+2\Delta)\sec\lambda\} \exp\{-|z|\}. \end{aligned}$$

Therefore, by the lemma, $f(z) = 0$.

But $f(z) \equiv f_1(x) - f_2(x)$; and therefore we have proved that $f_1(x) = f_2(x)$, when $f_1(x)$ and $f_2(x)$ are subject to the conditions stated at the beginning of the section.

The reader might be inclined to think, at first sight, that if $\beta-\alpha > 2\pi$, we could infer that $f_3(x)$ is identically zero on account of the theorem that "a non-convergent series cannot represent asymptotically the same one-valued analytic function for all arguments of x."†

This theorem is not applicable, because $f_1(x)$, $f_2(x)$ may not be analytic inside the circle $|x| = \gamma$; a multiform function may have an asymptotic expansion valid for a range of values of $\arg x$ greater than 2π.

Thus, the generalised hypergeometric function formula‡

$$\begin{aligned} x^{\alpha}\Gamma(\alpha)\Gamma(1-\rho)\,{}_1F_1(\alpha;\rho;x) + \Gamma(\alpha+1-\rho)\Gamma(\rho-1)\,x^{\alpha+1-\rho}\,{}_1F_1(\alpha-\rho+1;2-\rho;x) \\ = \Gamma(\alpha)\Gamma(\alpha+1-\rho)\,{}_2F_0\left(\alpha;\alpha+1-\rho;-\frac{1}{x}\right) \end{aligned}$$

is valid when $|\arg x| < \frac{3}{2}\pi$.

9. We can now show that if an analytic function, $f(x)$, possesses an asymptotic expansion for large values of $|x|$ with a grade and an outer grade *equal to unity*, the range of values of $\arg x$ over which the expansion is valid being greater than π, the function $f(x)$ is *absolutely summable* by the method of BOREL for a range of values of $\arg x$ just less than the range over which the expansion is valid,§ provided that BOREL's integral be taken along an appropriate path from 0 to ∞, not necessarily the real axis.

---

* The definition is possible since $\beta-\alpha > \pi l$.

† BROMWICH, 'Theory of Infinite Series,' p. 335.

‡ See BARNES, 'Cambridge Philosophical Transactions,' vol. 20, p. 260.

§ The range of validity being taken less than 2π.

Let us suppose that the expansion is valid when $|x| \geqq \gamma$ and $\alpha \leqq \arg x \leqq \beta$, where $\beta - \alpha = \pi + 2\lambda$ and $0 < \lambda < \frac{1}{2}\pi$.

Putting $z = x \exp\{-\frac{1}{2}(\alpha+\beta)i\}$ and $f(x) \equiv \mathrm{F}(z)$, we see that $\mathrm{F}(z)$ possesses an asymptotic expansion of the form*

$$\mathrm{F}(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_n}{z^n} + \mathrm{R}_n,$$

where

$$|a_n| \leqq \mathrm{A} \,.\, n!\,\rho^n, \qquad |\mathrm{R}_n| \leqq \mathrm{B} \,.\, n!\,\sigma^n |z|^{-n-1},$$

when $|\arg z| \leqq \frac{1}{2}\pi + \lambda$ and $|z| \geqq \gamma$.

We notice that

$$|\mathrm{R}_n| \leqq \mathrm{J}_n |x|^{-n},$$

where $\mathrm{J}_n = \mathrm{B} \,.\, n!\,\sigma^n \gamma^{-1}$.

Let the region in which the asymptotic expansion of $\mathrm{F}(z)$ is valid be called the region D.

Let L be a contour formed by the following lines :—

(i) The portion of the ray $\arg z = -(\frac{1}{2}\pi + \theta)$ for which $|z| \geqq \gamma|t|$, θ being an arbitrary quantity, as small as we please, such that $0 < \theta < \lambda$.

(ii) The major arc of the circle $|z| = \gamma|t|$ terminated by the points $\gamma|t| \exp\{\pm(\frac{1}{2}\pi + \theta)i\}$.

(iii) The portion of the ray $\arg z = \frac{1}{2}\pi + \theta$ for which $|z| \geqq \gamma|t|$.

Let us consider the function $\phi(t)$ defined by the equation

$$\frac{1}{2\pi i}\int_{\mathrm{L}} \mathrm{F}(z/t) \,.\, z^{-1} e^z \, dz = \phi(t),$$

where $|\arg t| \leqq \lambda - \theta$.

We observe that when z is at any point on the contour L, z/t lies within or on the boundary of the region D.

Now let us define functions $\psi_1, \psi_2, \ldots, \psi_n, \ldots$, by the system of equations

$$\int_v^\infty \frac{\mathrm{F}(u) - a_0}{u} \, du = \psi_1(v), \qquad \int_v^\infty \left\{\psi_n(u) - \frac{a_n}{n!\,u}\right\} du = \psi_{n+1}(v) \quad . \quad . \quad . \qquad (15)$$

The path of integration is supposed to be taken along the ray from v to infinity which, when produced backwards, passes through the point $u = 0$; we deduce by continued integration that the asymptotic expansion of $\psi_n(v)$ is

$$\psi_n(v) = \frac{a_n}{n!\,v} + \frac{1!\,a_{n+1}}{(n+1)!\,v^2} + \ldots + \frac{m!\,a_{n+m}}{(n+m)!\,v^m} + \frac{m!\,\mathrm{P}_{n+m}}{(n+m)!\,v^m}$$

where

$$|\mathrm{P}_{n+m}| \leqq |\mathrm{J}_{n+m}| \leqq \mathrm{B} \,.\, (n+m)!\,\sigma^{n+m}\gamma^{-1},$$

provided that v lies within the region D or on its boundary.

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* This function F will not be confused with the F of Section 7.

Also, we notice that, by CAUCHY's theorem,

$$\frac{1}{2\pi i}\int_{\mathrm{L}} z^{-n-1}e^{z}\,dz = \text{the residue of } z^{-n-1}e^{z} \text{ at the origin} = \frac{1}{n!}\cdot$$

By making use of the equations (15) in conjunction with this last result we get in succession on integrating by parts :

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathrm{L}} \mathrm{F}(z/t)\,.\,z^{-1}e^{z}\,dz \\
&\quad = \frac{1}{2\pi i}\int_{\mathrm{L}} a_0 z^{-1}e^{z}\,dz - \frac{1}{2\pi i}\int_{\mathrm{L}}\left\{\frac{d}{dz}\psi_1(z/t)\right\}e^{z}\,dz, \\
&\quad = a_0 - \frac{1}{2\pi i}\left[e^{z}\,.\,\psi_1(z/t)\right]_{\mathrm{L}} + \frac{1}{2\pi i}\int_{\mathrm{L}}\psi_1(z/t)\,.\,e^{z}\,dz, \\
&\quad = a_0 + \frac{1}{2\pi i}\int_{\mathrm{L}}\psi_1(z/t)\,e^{z}\,dz, \\
&\quad = a_0 + \frac{1}{2\pi i}\int_{\mathrm{L}}\left\{a_1\frac{t}{z} - t\frac{d}{dz}\psi_2(z/t)\right\}e^{z}\,dt, \\
&\quad = a_0 + \frac{a_1 t}{1!} - \frac{t}{2\pi i}\left[\psi_2(z/t)\,.\,e^{z}\right]_{\mathrm{L}} + \frac{t}{2\pi i}\int_{\mathrm{L}}\psi_2(z/t)\,.\,e^{z}\,dz, \\
&\quad = a_0 + \frac{a_1 t}{1!} + \frac{t}{2\pi i}\int_{\mathrm{L}}\psi_2(z/t)\,.\,e^{z}\,dz, \\
&\quad = \ldots\ldots\ldots\ldots, \\
&\quad = a_0 + \frac{a_1 t}{1!} + \frac{a_2 t^2}{2!} + \ldots + \frac{a_n t^n}{n!} + \frac{t^n}{2\pi i}\int_{\mathrm{L}}\psi_{n+1}(z/t)\,.\,e^{z}\,dz, \quad \ldots\ldots \quad (16)
\end{aligned}$$

each of the terms in square brackets vanishing at both ends of the contour.

We shall now estimate the value of $\int_{\mathrm{L}}\psi_{n+1}(z/t)\,e^{z}\,dz$ for any value of n.

At all points on L we have

$$|\psi_n(z/t)| < \left|\frac{a_n}{n!\,z/t} + \frac{\mathrm{P}_n}{n!\,z/t}\right|,$$

i.e.,

$$|\psi_n(z/t)| < \mathrm{A}\rho^n\gamma^{-1} + \mathrm{B}\sigma^n\gamma^{-2}.$$

For brevity we put

$$\mathrm{A}\rho^n\gamma^{-1} + \mathrm{B}\sigma^n\gamma^{-2} = u_n.$$

On the arc of the circle we have

$$|\exp z| \leq \exp|\gamma t|,$$

and

$$dz = |\gamma t|\,e^{i\omega}i\,d\omega,$$

where ω varies from $-\frac{1}{2}\pi-\theta$ to $\frac{1}{2}\pi+\theta$.

Consequently, for the integral round the arc,

$$\left|\int \psi_{n+1}(z/t)\, e^z\, dz\right| < (\pi+2\theta)\, u_{n+1} |\gamma t| \exp |\gamma t|.$$

On the ray $\arg z = \frac{1}{2}\pi+\theta$, we put

$$z = |\gamma t| \,.\, r \exp\left(\tfrac{1}{2}\pi+\theta\right) i,$$

where r varies from 1 to ∞; so that for the integral along this ray,

$$\left|\int \psi_{n+1}(z/t)\,.\, e^z\, dz\right| < u_{n+1} \operatorname{cosec}\theta \exp\{-|\gamma t| \sin\theta\};$$

and we get the same inequality for the integral along the other ray of L.

Combining our results we get

$$\left|\frac{t^n}{2\pi i}\int_{\mathrm{L}} \psi_{n+1}(z/t)\, e^z\, dz\right|$$
$$< \frac{1}{2\pi}\left[(\pi+2\theta)\,\gamma\, |t| \exp|\gamma t| + 2 \operatorname{cosec}\theta \exp\{-|\gamma t| \sin\theta\}\right] \times \left[\mathrm{A}\gamma^{-1} |\rho t|^n + \mathrm{B}\gamma^{-2} |\sigma t|^n\right] \quad . \;.\; .\; .\; .\; . \;(16\text{A}).$$

Remembering that $\rho \leq \sigma$, we see from this formula that, if $|t| < \sigma^{-1}$, then

$$\left|\frac{t^n}{2\pi i}\int_{\mathrm{L}} \psi_{n+1}(z/t)\, e^z\, dz\right| \to 0$$

uniformly as $n \to \infty$.

That is to say, when $|\arg t| \leq \lambda-\theta$ and $|t| < \sigma^{-1}$, *we may expand the integral*

$$\frac{1}{2\pi i}\int_{\mathrm{L}} \mathrm{F}(z/t)\,.\, z^{-1} e^z\, dz$$

in the form

$$\frac{1}{2\pi i}\int_{\mathrm{L}} \mathrm{F}(z/t)\,.\, z^{-1} e^z\, dz = a_0 + \frac{a_1 t}{1\,!} + \frac{a_2 t^2}{2\,!} + \ldots + \frac{a_n t^n}{n\,!} + \Omega_n,$$

where $\Omega_n \to 0$ *as* $n \to \infty$.

In other words, when $|\arg t| \leq \lambda-\theta$ and $|t| < \sigma^{-1}$, the integral

$$\frac{1}{2\pi i}\int_{\mathrm{L}} \mathrm{F}(z/t)\,.\, z^{-1} e^z\, dz$$

represents BOREL's associated function defined by the series

$$\phi(t) = a_0 + \frac{a_1 t}{1\,!} + \frac{a_2 t^2}{2\,!} + \ldots$$

converging when $|t| < \rho^{-1}$; so that BOREL's associated function is analytic when $|t| < \rho^{-1}$.

Now the integral $$\frac{1}{2\pi i}\int_{\mathrm{L}} \mathrm{F}(z/t)\,.\, z^{-1} e^z\, dz$$

converges when $|\arg t| \leq \lambda-\theta$; and by putting $z = tu$, we find that the integral is an analytic function* of t in the region $|\arg t| < \lambda-\theta$.

That is to say, BOREL'S *associated function* $\phi(t)$ *is analytic in the region* $|t| < \rho^{-1}$, *and also in the region* $|\arg t| < \lambda-\theta$.

We shall require an upper limit for $\left|\dfrac{d^n\phi(t)}{dt^n}\right|$ when $|\arg t| < \lambda-\theta$.

To obtain this upper limit we consider the integral

$$\frac{1}{2\pi i}\int_L \frac{e^z}{z} \cdot \frac{z^n}{t^n}\left\{F\left(\frac{z}{t}\right)-a_0-a_1\frac{t}{z}- \ldots -a_{n-1}\frac{t^{n-1}}{z^{n-1}}\right\}dz.$$

We may show that it can be expanded in the form

$$a_n+\frac{a_{n+1}}{1\,!}t+\frac{a_{n+2}}{2\,!}t^2+ \ldots$$

when $|t| < \sigma^{-1}$, by replacing $F(z/t)$ in the work immediately preceding by

$$\frac{z^n}{t^n}\left\{F\left(\frac{z}{t}\right)-a_0-a_1\frac{t}{z}- \ldots -a_{n-1}\frac{t^{n-1}}{z^{n-1}}\right\}.$$

Therefore, by the theory of analytic continuation, we have

$$\frac{1}{2\pi i}\int_L \frac{e^z}{z} \cdot \frac{z^n}{t^n}\left\{F\left(\frac{z}{t}\right)-a_0-a_1\frac{t}{z}- \ldots -a_{n-1}\frac{t^{n-1}}{z^{n-1}}\right\}dz = \frac{d^n\phi(t)}{dt^n}$$

for all values of $|t|$ in the interior of the region $|\arg t| < \lambda-\theta$.

Now

$$\left|F\left(\frac{z}{t}\right)-a_0-a_1\frac{t}{z}- \ldots -a_{n-1}\frac{t^{n-1}}{z^{n-1}}\right| < B\,.\,(n-1)\,!\,\sigma^{n-1}\left|\frac{t}{z}\right|^n \qquad (n > 0)$$

$$< A+B\sigma\left|\frac{t}{z}\right| \qquad (n = 0),$$

so that

$$\left|\frac{z^n}{t^n}\left\{F\left(\frac{z}{t}\right)-a_0-a_1\frac{t}{z}- \ldots -a_{n-1}\frac{t^{n-1}}{z^{n-1}}\right\}\right| < B\,.\,(n-1)\,!\,\sigma^{n-1} \qquad (n > 0)$$

$$< A+B\sigma\gamma^{-1} \qquad (n = 0).$$

Making the substitutions made in obtaining (16A), we find that

$$\left.\begin{aligned}\left|\frac{d^n\phi(t)}{dt^n}\right| &< K\,.\,(n-1)\,!\,\sigma^{n-1}|t^{-1}|\exp|\gamma t| \qquad n > 0\\ |\phi(t)| &< K'|t^{-1}|\exp|\gamma t| \qquad n = 0\end{aligned}\right\}, \quad . \quad . \quad . \quad (17)$$

where K, K′ are finite quantities independent of t and n.

---

* This follows, without difficulty, from an obvious modification of the theorem stated by BROMWICH, 'Theory of Infinite Series,' p. 438.

We use the formulæ (17) when $|t|$ is greater than, say, $\frac{1}{4}\sigma^{-1}$.

When $|t| \leq \frac{1}{4}\sigma^{-1}$, since the series for $\phi(t)$ converges when $|t| < \rho^{-1}$ (and $\rho \leq \sigma$), we have $|\phi(t)| < \mathrm{K}''$ where K'' is finite and independent of t.

Also when $|t| \leq (\frac{1}{4}+\frac{2}{3})\,\sigma^{-1}$ we have $|\phi(t)| < \mathrm{K}'''$ where K''' is finite and independent of t.

Accordingly, when $|t| \leq \frac{1}{4}\sigma^{-1}$, we use the formula

$$\frac{d^n\phi(t)}{dt^n} = \frac{n!}{2\pi i}\int \frac{\phi(\xi)}{(\xi-t)^{n+1}}\,dt,$$

the contour of integration being a circle of radius $\frac{2}{3}\sigma^{-1}$.

Hence we get, when $|t| \leq \frac{1}{4}\sigma^{-1}$,

$$\left|\frac{d^n\phi(t)}{dt^n}\right| < \mathrm{K}'''\,(\tfrac{3}{2}\sigma)^n\,.\,n!.$$

Combining these results with (17), we see that for all values of t such that *either* $|\arg t| < \lambda-\theta$ *or* $|t| \leq \frac{1}{4}\sigma^{-1}$ we have

$$|\phi(t)| < \mathrm{K}_1 \exp|\gamma t|, \qquad \left|\frac{d^n\phi(t)}{dt^n}\right| < \mathrm{K}_2\,(\tfrac{3}{2}\sigma)^n\,.\,n!\exp(|\gamma t|) \quad . \quad . \quad (17\text{A})$$

where K_1, K_2, are finite and independent of t.

Consequently, if $\gamma_1 > \gamma$, and n is any assigned integer,

$$\underset{t\to\infty}{\mathrm{L}t}\left|\exp(-\gamma_1|t|)\,\frac{d^n\phi(t)}{dt^n}\right| = 0,$$

if $\gamma_1 > \gamma$; and the function $\phi(t)$ is analytic in the region in which *either* of the inequalities

$$\text{(i.)}\ |t| < \rho^{-1}, \qquad \text{(ii.)}\ |\arg t| < \lambda-\theta,$$

is satisfied.

Now let us study the function

$$\int_0^\infty z\phi(t)\,e^{-zt}\,dt = \mathrm{F}_1(z).$$

The function $\mathrm{F}_1(z)$ is an analytic function of z when $\mathrm{R}(z) > \gamma_1$ where $\gamma_1 > \gamma$.

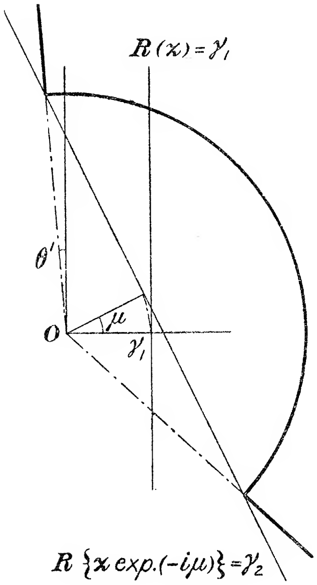
If also $\mathrm{R}\{z\exp(-i\mu)\} > \gamma_1$, we can see that

$$\int_{(-\mu)} z\phi(t)\,e^{-zt}\,dt = \mathrm{F}_1(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where μ is any quantity such that $0 < \mu < \lambda-\theta$, $\mu < \frac{1}{4}\pi$.

Equation (18) gives the analytic continuation of $\mathrm{F}_1(z)$ over the whole of the area for which $\mathrm{R}\{z\exp(-i\mu)\} > \gamma_1$.

Let θ' be a small quantity such that $0 < \theta' < \mu$.



Then $F_1(z)$ is certainly analytic in the region (see figure) in which both the inequalities

$$|z| > \gamma_1 \operatorname{cosec}(\mu-\theta'), \qquad -\tfrac{1}{2}\pi-\theta'+2\mu < \arg z < \tfrac{1}{2}\pi+\theta',$$

are satisfied.

In like manner, when

$$R(z) > \gamma_1,\ R\{z\exp(i\mu)\} > \gamma_1,$$

we have

$$\int_{(\mu)} z\phi(t)\,e^{-zt}\,dt = F_1(z),$$

and we can deduce that $F_1(z)$ is analytic in the region in which both the inequalities

$$|z| > \gamma_1 \operatorname{cosec}(\mu-\theta'), \qquad -\tfrac{1}{2}\pi-\theta' < \arg z < \tfrac{1}{2}\pi+\theta'-2\mu$$

are satisfied.

That is to say, $F_1(z)$ is analytic in the region in which the inequalities

$$|z| > \gamma_1 \operatorname{cosec}(\mu-\theta'), \qquad |\arg z| < \tfrac{1}{2}\pi+\theta',$$

are satisfied.

Now consider the function $F_1(z)$ in the region in which

$$|z| > (\gamma_1+1) \operatorname{cosec}(\mu-\theta'), \qquad |\arg z| < \tfrac{1}{2}\pi+\theta'.$$

We may define $F_1(z)$ in this region by the equation

$$F_1(z) = \int_{(\mp\mu)} z\phi(t)\,e^{-zt}\,dt,$$

where the upper sign may be taken if $\arg z \geqq 0$, and the lower sign if $\arg z \leqq 0$.

By repeated integration by parts we get

$$F_1(z) = \left[-\phi(t)\,e^{-zt}\right]_{(\mp\mu)} + \left[-\frac{\phi'(t)\,e^{-zt}}{z}\right]_{(\mp\mu)} + \ldots + \left[-\frac{\phi^{(n)}(t)\,e^{-zt}}{z^n}\right]_{(\mp\mu)} + \int_{(\mp\mu)} \frac{\phi^{(n+1)}(t)\,e^{-zt}}{z^n}\,dt.$$

From the results (17A) all the integrated parts vanish at the upper limit; and we have

$$F_1(z) = a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + S_n$$

where

$$|S_n| = \left|\int_{(\mp\mu)} \frac{\phi^{(n+1)}(t)\,e^{-zt}}{z^n}\,dt\right|.$$

Now $\gamma_1|t|-\mathrm{R}(zt) \leq -|t|$; and hence from (17A)

$$|\mathrm{S}_n z^n| < \mathrm{K}_2(\tfrac{3}{2}\sigma)^{n+1}(n+1)!\int_{(\mp\mu)}\exp(-|t|)\,d|t|,$$

$$< \mathrm{K}_2(\tfrac{3}{2}\sigma)^{n+1}(n+1)!.$$

and K_2 is independent of n.

But we have $|\mathrm{S}_n| = \left|\frac{a_{n+1}}{z^{n+1}}+\mathrm{S}_{n+1}\right|$; so that, applying the formula just obtained to S_{n+1}, we get

$$|\mathrm{S}_n z^{n+1}| < \mathrm{A}\rho^{n+1}(n+1)!+\mathrm{K}_2(\tfrac{3}{2}\sigma)^{n+1}.(n+2)!.$$

But we can find a finite quantity K independent of n such that $\frac{(n+1)(n+2)}{(4/3)^n} < \mathrm{K}$ when n is a positive integer; and *a fortiori* $\frac{n+1}{2^n} < \mathrm{K}$; therefore, since $\rho \leq \sigma$,

$$|\mathrm{S}_n z^{n+1}| < \mathrm{K}\{\mathrm{A}\rho+\mathrm{K}_2.\tfrac{3}{2}\sigma\}(2\sigma)^n.n! < \mathrm{B}_2.(2\sigma)^n.n!, \text{ say.}$$

That is to say, if $|z| > (\gamma_1+1)\operatorname{cosec}(\mu-\theta')$, $|\arg z| < \frac{1}{2}\pi+\theta'$, we have asymptotic expansions of the form

$$\left.\begin{aligned} \mathrm{F}(z) &= a_0+\frac{a_1}{z}+\ldots+\frac{a_n}{z^n}+\mathrm{R}_n, \\ \mathrm{F}_1(z) &= a_0+\frac{a_1}{z}+\ldots+\frac{a_n}{z^n}+\mathrm{S}_n, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where

$$|\mathrm{R}_n.z^{n+1}| < \mathrm{B}.n!\,\sigma^n, \qquad |\mathrm{S}_n.z^{n+1}| < \mathrm{B}_2.n!\,(2\sigma)^n. \quad . \quad . \quad . \quad (19\mathrm{A})$$

Taking $l = 1$, and writing 2σ for σ in Theorem V., we conclude, since the expansions (19) are valid when $|\arg z| \leq \frac{1}{2}\pi+\frac{1}{2}\theta'$ (*i.e.*, for a range of values of $\arg z$ greater than π), that

$$\mathrm{F}(z) = \mathrm{F}_1(z).$$

That is to say, in the region $|z| > (\gamma_1+1)\operatorname{cosec}(\mu-\theta')$, $|\arg z| < \frac{1}{2}\pi+\theta'$, we have proved that

$$\mathrm{F}(z) = \int_{(\mp\mu)} z\phi(t)\,e^{-zt}\,dt.$$

But $\int_{(-\mu)} z\phi(t)\,e^{-zt}\,dt$ is an analytic function of z when $\mathrm{R}\{z\exp(-i\mu)\} > \gamma_1$, where γ_1 is any quantity greater than γ; hence, by the theory of analytic continuation, we have

$$\mathrm{F}(z) = \int_{(-\mu)} z\phi(t)\,e^{-zt}\,dt;$$

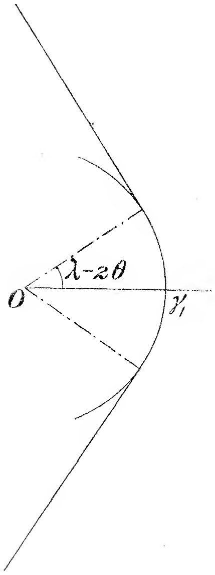
and more generally, if $-(\lambda-\theta) < \nu < \lambda-\theta$, we have*

$$\mathrm{F}(z) = \int_{(-\nu)} z\phi(t)\,e^{-zt}\,dt, \quad \text{provided } \mathrm{R}\{z\exp(-i\nu)\} > \gamma_1.$$

---

* $\phi(t)$ is analytic along the ray $\arg t = -\nu$.

Now draw the circle $|z| = \gamma_1$; and draw the tangents to this circle at $z = \gamma_1 \exp\{\pm i(\lambda - 2\theta)\}$ in the directions of the rays $\arg z = \pm\{\frac{1}{2}\pi + \lambda - 2\theta\}$ respectively.



If z be any point to the right of the curve formed by these tangents and the arc of the circle joining their extremities (see figure), we can find a real quantity ν such that $|\nu| < \lambda - \theta$, and such that

$$\mathrm{R}\{z \exp(-i\nu)\} > \gamma_1.$$

For such a value of z we have "summed" F(z) by the equation which we have just proved, viz.:

$$\mathrm{F}(z) = \int_{(-\nu)} z\phi(t)\, e^{-zt}\, dt.$$

In other words, we have shown that F(z) *is summable by means of an integral of the same nature as* **Borel's** *integral.*[*]

Returning to the equations at the beginning of the section,

$$z = x \exp\{-\tfrac{1}{2}(\alpha+\beta)\, i\}, \qquad f(x) = \mathrm{F}(z),$$

we see that $f(x)$ is summable by an integral of the same nature as Borel's integral; the formal result is hardly worth writing down, since it usually happens that $\alpha = -\beta$, so that $z = x$.

10. We may also show, by the methods of Section 9, that if we are given a function $\phi(t)$ defined by the series

$$\phi(t) = a_0 + \frac{a_1 t}{1!} + \frac{a_2 t^2}{2!} + \ldots,$$

where $|a_n| < \mathrm{A}\,.\,n!\,\rho^n$, and if $\phi(t)$ have no singularities in the region $|\arg t| \leqq \lambda$, and if when $|\arg t| \leqq \lambda$, $|\phi(t)| < \mathrm{K} \exp\{\gamma |t|\}$, where K is a constant, then the function F(z) defined by the equation

$$\mathrm{F}(z) = \int_0^\infty z\phi(t)\, e^{-zt}\, dt$$

has an asymptotic expansion in powers of $1/z$ valid when $|\arg z| < \frac{1}{2}\pi + \lambda - \theta$ (where $\theta > 0$), provided that $|z|$ be sufficiently large; and that unity is a grade and outer grade of the expansion, and ρ is a radius.

For, when $|\arg t| \leqq \lambda - \theta$, we have

$$\phi^n(t) = \frac{n!}{2\pi i} \int \frac{\phi(\xi)}{(\xi - t)^{n+1}}\, d\xi,$$

the contour being a circle of radius $\rho_1^{-1} \sin\theta$, where $\rho_1 > \rho$; assuming that when $|t| \leqq \rho_1^{-1}$, $|\phi(t)| < \mathrm{K}$, we find without difficulty that

$$|\phi_n(t)| < n!\,\mathrm{K}\,[\exp\{\gamma|t| + \gamma\rho_1^{-1}\sin\theta\}]\,\{\rho_1 \operatorname{cosec}\theta\}^n.$$

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[*] From the results proved concerning $\phi^{(n)}(t)$ it follows that F(z) is "absolutely summable."

Now suppose that we can find a real quantity ν such that $|\nu| \leqq \lambda - \theta$, $R\{z \exp(-i\nu)\} > \gamma + 1$; then we may write

$$F(z) = \int_{(-\nu)} z\phi(t)\, e^{-zt}\, dt;$$

and, on integrating by parts, we get

$$F(z) = a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + \int_{(-\nu)} \frac{\phi^{(n+1)}(t)\, e^{-zt}}{z^n}\, dt,$$

i.e.,

$$F(z) = a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + R_n,$$

where

$$|R_n z^n| < (n+1)!\, K \{\exp(\gamma\rho_1^{-1} \sin\theta)\} \{\rho_1 \operatorname{cosec}\theta\}^n \int_{(-\nu)} \exp\{-|t|\}\, d|t|,$$

i.e.,

$$|R_n z^n| < (n+1)!\, K' (\rho_1 \operatorname{cosec}\theta)^n, \quad \text{where } K' \text{ is finite.}$$

Now

$$R_n = \frac{a_{n+1}}{z^{n+1}} + R_{n+1},$$

so that

$$|R_n| < \{A\rho^{n+1} . (n+1)! + (n+2)!\, K' (\rho_1 \operatorname{cosec}\theta)^n\} |z|^{-n-1}.$$

If $\rho_2 > \rho_1$, we have

$$\rho^n (n+1) < K''\rho_2^n, \qquad \rho_1^n (n+1)(n+2) < K'''\rho_2^n,$$

where K'', K''' are finite and independent of n.

Therefore

$$|R_n| < B(\rho_2 \operatorname{cosec}\theta)^n . n!\, |z|^{-n-1}.$$

That is to say, for values of z such that

$$R\{z \exp(-i\nu)\} > \gamma + 1,$$

where $|\nu|$ is less than or equal to $\lambda - \theta$, $F(z)$ has an asymptotic expansion with grades equal to unity, a radius ρ, and an outer radius $\rho_2 \operatorname{cosec}\theta$, where ρ_2 is any quantity greater than ρ. This is, effectively, the result stated at the beginning of the section.

11. We shall conclude by investigating the characteristics of the asymptotic expansion of a function connected with the "logarithmic integral," or "*li*" function, defined by the equation

$$li(e^{-x}) = \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

When x is real and positive it is known that*

$$e^{x}\,li\,(e^{-x}) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} + \ldots + (-)^{n-1}\frac{(n-1)!}{x^n} + \mathrm{R}_n,$$

where $|\mathrm{R}_n| < n!\,x^{-n-1}$.

Thus, when x is real and positive, $e^{x}\,li\,(e^{-x})$ has an asymptotic expansion of which characteristics are

$$k = 1, \qquad \rho = 1,$$
$$l = 1, \qquad \sigma = 1.$$

Suppose that x is complex, but not a real negative quantity. Then we may prove that

$$\begin{aligned} e^{x}\,li\,(e^{-x}) &= \int_0^\infty \frac{e^{-v}}{x+v}\,dv \\ &= \int_0^\infty e^{-v}\left\{\frac{1}{x} - \frac{v}{x^2} + \ldots + (-)^{n-1}\frac{v^{n-1}}{x^n}\right\} dv + (-)^n \int_0^\infty \frac{v^n e^{-v}}{x^n\,(x+v)}\,dv, \end{aligned}$$

so that

$$e^{x}\,li\,(e^{-x}) = \frac{1}{x} - \frac{1!}{x^2} + \ldots + (-)^{n-1}\frac{(n-1)!}{x^n} + \mathrm{R}_n,$$

where

$$|\mathrm{R}_n| \leq \int_0^\infty \frac{v^n e^{-v}}{|x^n\,(x+v)|}\,dv.$$

If $\mathrm{R}(x) \geq 0$, on the path of integration $|x+v| \geq |x|$.

If $\mathrm{R}(x) \leq 0$, then $|x+v| \geq |\mathrm{I}(x)|$.

Thus, if $\mathrm{R}(x) \geq 0$,

$$|\mathrm{R}_n| \leq |x|^{-n-1}\int_0^\infty v^n e^{-v}\,dv \leq |x|^{-n-1}\,.\,n!\,.$$

Whereas, if

$$\tfrac{1}{2}\pi \leq |\arg x| \leq \tfrac{1}{2}\pi + \alpha \qquad (\alpha < \tfrac{1}{2}\pi),$$

we have

$$|\mathrm{I}(x)| \geq |x|\cos\alpha,$$

so that

$$|\mathrm{R}_n| \leq \sec\alpha\,.\,|x|^{-n-1}\,.\,n!\,.$$

Thus, if $|\arg x| \leq \frac{1}{2}\pi + \alpha$, the function $e^{x}\,li\,(e^{-x})$ possesses an asymptotic expansion of which characteristics and constants are

$$k = 1, \qquad \rho = 1, \qquad \mathrm{A} = 1,$$
$$l = 1, \qquad \sigma = 1, \qquad \mathrm{B} = 1 \text{ or } \sec\alpha,$$

the value unity being taken for B if $|\arg x| \leq \frac{1}{2}\pi$.

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* WHITTAKER, 'Modern Analysis,' § 87.

Other functions which may be investigated in a similar manner are to be found in BROMWICH'S 'Theory of Infinite Series,' pp. 351–352.

The investigation of the characteristics of asymptotic series representing functions defined by integrals of types essentially different from BOREL'S type is to be found in the paper by the writer, cited on p. 282.

12. Finally, it may be stated that it is possible to conceive a function which has an asymptotic expansion

$$a_0 + \frac{a_1}{x} + \ldots + \frac{a_n}{x^n} + R_n,$$

wherein, as $n \to \infty$, $|a_n|$ increases more rapidly than any expression of the form $A\Gamma(kn+1)\rho^n$; such an expansion would be obtained by taking $a_n = G(n+1)$ where $G(n)$ is BARNES' G-function. But such series have not yet occurred in analysis, and they do not appear to possess the interesting properties of series with finite characteristics.